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Effective Impedance of a Statistically Rough Sphere: I. A General Case

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Using Bourret's approximation, the mean electromagnetic field scattered by a statistically rough sphere having a small surface impedance η_0 is analysed. Reflection coefficients of the mean field are calculated, which are expressed in terms of effective impedances of spherical multipole waves. In contrast to the previous research of the authors, not only the disturbances proportional to the variance σ^2 of irregularity heights, but also those proportional to $\eta_0\sigma^2$ are taken into account , which proves necessary when η_0 is not very small.

Introduction

The pronounced development of the perturbation theory for the problem of wave scattering by a statistically rough sphere began rather recently [1-4]. This is explained by the complexity of the mathematical formalism involved that is based on the theory of representations of the sphere rotation group [5,6] and is non-traditional for this field of research. In papers [1,2], study of the incoherent scattering in Born's approximation for inversion of the random scattering operator is carried out.

Papers [3,4] contain, in general, results for the coherent field in Bourret's approximation and, in particular, effective impedances of spherical waves of electric and magnetic multipoles in which terms the mean field is expanded in a series .

The expansion of boundary conditions in [3,4] is carried out up to perturbing terms quadratic in the height of irregularities σ^2. Neglecting the perturbation terms of order $\sim\eta_0\sigma^2$, where η_0 is the surface impedance of an unperturbed smooth sphere $(|\eta_0|<<1)$, can greatly facilitate the analysis, however at the expense of limitations on the magnitude of η_0 and the form of irregularity spectrum, even in the case of wave scattering by a rough plane [7]. A natural continuation of the research begun in [3,4] is the study of scattering by a rough sphere with account of all perturbing terms of $\sim\sigma^2$, including $\sim\eta_0\sigma^2$.

Effective reflection coeffitients for spherical waves

We shall proceed from Leontovich's boundary conditions on a statistically rough sphere $r=a+\zeta(\varphi,\theta)$ with a small surface impedance η_0:

$$\left[\vec{N}\times\vec{E}\right]=\eta_0\left[\vec{N}\times\left[\vec{N}\times\vec{H}\right]\right]\Big|_{r=a+\zeta(\varphi,\theta)}. \qquad (1)$$

Here $\vec{N}=\left(\hat{i}_r-\vec{\gamma}\right)/\sqrt{1+\gamma^2}$ is an exterior normal to the surface, $r=a+\zeta(\varphi,\theta)$, $\vec{\gamma}=\nabla\zeta(\varphi,\theta)$; $\vec{E}$ and $\vec{H}$ are the electric and magnetic field vectors. Irregularities of the surface ζ are supposed to be small and flat, $|\zeta|/a, k|\zeta|<<1$, $|\nabla\zeta|<<1$; $\zeta(\varphi,\theta)-$ being a stochastic function with zero mean value $\langle\zeta\rangle=0$, whose variance is equal to $\sigma^2=\langle\zeta^2\rangle$. These assumptions allow to apply the perturbation theory in the boundary conditions (1). For this aim, equation (1) ought to be multiplied by a unit vector $\hat{i}_r$ of the normal to the mean surface $r=a$ and to expand $\vec{E}(\vec{r}),\vec{H}(\vec{r})$ and $\vec{H}(\vec{r})$ close to the surface $r=a$ in a series in terms of powers of ζ and γ to within quadratic terms. After splitting the fields $\vec{E}$ and $\vec{H}$ into mean, $\vec{\mathcal{E}}=\langle\vec{E}\rangle$ and $\vec{\mathcal{H}}=\langle\vec{H}\rangle$, and fluctuating, $\vec{e}$ and $\vec{h}$, components, equation (1) will be transformed into a set of two coupled "equivalent" boundary conditions on the mean surface $r=a$, viz.

$$\begin{cases} \left(1+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2}\right)\vec{E}_\perp - \left(\eta_0/\sqrt{2}\right)\left(1+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2}\right)\left\{\hat{i}_-\left(H_\theta - iH_\varphi\right) - \right. \\ \left. -\hat{i}_+\left(H_\theta + iH_\varphi\right)\right\}\Big|_{r=a} = -\vec{U}_1 - \vec{U}_2 + \vec{U}_3 + \vec{U}_4 - \vec{U}_5 + \vec{U}_6 = -\vec{U}, \qquad (2a) \\ \vec{e}_\perp - \left(\eta_0/\sqrt{2}\right)\left\{\hat{i}_-\left(h_\theta - ih_\varphi\right) - \hat{i}_+\left(h_\theta + ih_\varphi\right)\right\}\Big|_{r=a} = -\vec{u}_1 - \vec{u}_2 + \vec{u}_3 + \vec{u}_4 = -\vec{u}, \qquad (2b) \end{cases}$$

where

$$\vec{U}_1 = \left\langle \zeta\frac{\partial}{\partial r}\vec{e}_\perp\right\rangle,\quad \vec{U}_2 = \left\langle \vec{\gamma}\, e_r\right\rangle,$$

$$\vec{U}_3 = \left(\eta_0/\sqrt{2}\right)\times$$
$$\times\left\langle \hat{i}_-\zeta\frac{\partial}{\partial r}\left(h_\theta - ih_\varphi\right) - \hat{i}_+\zeta\frac{\partial}{\partial r}\left(h_\theta + ih_\varphi\right)\right\rangle,$$

$$\vec{U}_4 = \left(\eta_0/\sqrt{2}\right)\left\langle \hat{i}_- h_r\left(\gamma_\theta - i\gamma_\varphi\right) - \hat{i}_+ h_r\left(\gamma_\theta + i\gamma_\varphi\right)\right\rangle,$$

$$\vec{U}_5 = \left(\eta_0/\sqrt{2}\right)\times$$
$$\times\left\langle\left\{\hat{i}_-\left(\gamma_\theta - i\gamma_\varphi\right) - \hat{i}_+\left(\gamma_\theta + i\gamma_\varphi\right)\right\}\left(\mathcal{H}_\theta\gamma_\theta + \mathcal{H}_\varphi\gamma_\varphi\right)\right\rangle,$$

$$\vec{U}_6 = \left(\eta_0/\sqrt{2}\right)\frac{\gamma_0^2}{2}\left\langle \hat{i}_-\left(\mathcal{H}_\theta - i\mathcal{H}_\varphi\right) - \hat{i}_+\left(\mathcal{H}_\theta + i\mathcal{H}_\varphi\right)\right\rangle,$$

$$\vec{u}_1 = \zeta\frac{\partial}{\partial r}\vec{\mathcal{E}}_\perp,\quad \vec{u}_2 = \vec{\gamma}\,\mathcal{E}_r,$$

$$\vec{u}_3 = \left(\eta_0/\sqrt{2}\right)\times$$
$$\times\left\{\hat{i}_-\zeta\frac{\partial}{\partial r}\left(\mathcal{H}_\theta - i\mathcal{H}_\varphi\right) - \hat{i}_+\zeta\frac{\partial}{\partial r}\left(\mathcal{H}_\theta + i\mathcal{H}_\varphi\right)\right\},$$

$$\vec{u}_4 = \left(\eta_0/\sqrt{2}\right)\left\{\hat{i}_-\left(\gamma_\theta - i\gamma_\varphi\right)\mathcal{H}_r - \hat{i}_+\left(\gamma_\theta + i\gamma_\varphi\right)\mathcal{H}_r\right\},$$

$\gamma_0^2 = \langle\gamma^2\rangle$. The subscript $\perp$ designates vector components lying within the plane tangent to the mean surface $r = a$, the time dependence is $e^{-i\omega t}$. Thereby, for the reason stated below, instead of the natural spherical basis $\hat{i}_r$, $\hat{i}_\theta$ and $\hat{i}_\varphi$, the spiral one of $\hat{i}_0$, $\hat{i}_+$, $\hat{i}_-$ is used [6] (See Appendix). Note that the unit vectors of the spiral basis $\hat{i}_+$, $\hat{i}_-$ are, apart from the factor $(-i)$, contravariant basic vectors. The complex conjugation used below converts the contravariant basis into a covariant one ([6], Chapter 1).

The equivalent boundary conditions in [4] did not take into account the summands $\vec{U}_3, \vec{U}_4, \vec{U}_5, \vec{U}_6, \vec{u}_3$, and $\vec{u}_4$, nor $\sigma^2\cdot\partial^2/\partial r^2$ in equations (2a, 2b). Usually, the field vectors in a spherical problem are represented by expansions

$$\vec{\mathcal{E}} = -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(A_{nm}^{(e,o)}\vec{m}_{nm}^{(e,o)} + B_{nm}^{(e,o)}\vec{n}_{nm}^{(e,o)}\right), \qquad (3)$$

$$\vec{\mathcal{H}} = \frac{ik}{\omega\mu}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(A_{nm}^{(e,o)}\vec{n}_{nm}^{(e,o)} + B_{nm}^{(e,o)}\vec{m}_{nm}^{(e,o)}\right), \qquad (4)$$

where the indices "e" and "o" correspond, respectively, to the even and odd part of the appropriate potentials [8]. Coefficients of the expansions in similar series for $\vec{e}$, $\vec{h}$ will be designated $a_{nm}^{(e,o)}$ and $b_{nm}^{(e,o)}$. The vectorial wave harmonics $\vec{m}_{nm}^{(e,o)}$ and $\vec{n}_{nm}^{(e,o)}$ depend on spherical Bessel functions $Z_n(kr)$, on associated Legendre functions $P_n^m(\cos\theta)$ and on trigonometrical functions $\sin m\varphi, \cos m\varphi$. Up to a certain stage of the solution, the expressions $A_{nm}^{(e,o)}Z_n(kr)$, $B_{nm}^{(e,o)}Z_n(kr)$ are necessary to be understood as admissible linear combinations of spherical waves which are running to the sphere and from it. In expressions for $\vec{e}$, $\vec{h}$, according to the Rayleigh's hypothesis, only one type of waves, namely, ones running from the sphere, is possible. The expansion in such form is not invariant in relation to rotations of the sphere. The transition to the invariant form of expansion allows, in the most successful way, to use a method of small disturbances in the solution of the problem. For the transform to the invariant form of expansion, in addition to introduction of spiral basis, it is necessary to proceed from usual spherical functions to generalized ones (Wigner's functions)

$$t_{m,m'}^{n}\left(-\varphi,\theta,0\right) = e^{im\varphi}P_{m,m'}^{n}\left(\cos\theta\right)e^{im'0}, \qquad (5)$$

of order (or weight) of n, the functions being dependent on Euler angles φ,θ and $\psi = 0$ [5]. As a result, the vectorial wave harmonics $\vec{m}_{nm}^{(e,o)}$ and $\vec{n}_{nm}^{(e,o)}$

are transformed to linear combinations of the following vectorial functions

$$\begin{cases} \hat{i}^{+}_{nm} = \sqrt{\dfrac{2n+1}{4\pi}}\, t^{n}_{m,-1}(-\varphi,\theta,0)\cdot \hat{i}_{+}, \\ \hat{i}^{-}_{nm} = \sqrt{\dfrac{2n+1}{4\pi}}\, t^{n}_{m,+1}(-\varphi,\theta,0)\cdot \hat{i}_{-}, \\ \hat{i}^{0}_{nm} = \sqrt{\dfrac{2n+1}{4\pi}}\, t^{n}_{m,0}(-\varphi,\theta,0)\cdot \hat{i}_{0}, \end{cases} \quad (6)$$

$(n=0,1,2,\ldots; \quad -n \le m \le n)$.

Here, $N_{nm} = \sqrt{\dfrac{2n+1}{4\pi}}$ – is a normalizing factor. Orthogonality and completeness of the system of functions (6) is stated, for example, in [6], Chapter 4.

It is supposed that the stochastic field $\zeta(\varphi,\theta)$ has a spectral representation

$$\zeta(\varphi,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \zeta^{(e,o)}_{nm} P^{m}_{n}(\cos\theta)\begin{pmatrix}\cos m\varphi \\ \sin m\varphi\end{pmatrix} =$$

$$= \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \tilde{\zeta}^{(e,o)}_{nm}\, t^{n}_{m,0}(-\varphi,\theta,0), \quad (7)$$

$$\tilde{\zeta}^{(e,o)}_{nm} = \frac{-2i}{\sqrt{n(n+1)}}\zeta^{(e,o)}_{n|m|} Q^{(e,o)}_{nm},$$

$$Q^{(e,o)}_{nm} = \frac{1}{2} i^{|m|+1}\sqrt{n(n+1)\frac{(n+|m|)!}{(n-|m|)!}}\,\chi^{(e,o)}_{m},$$

$$\chi^{(e,o)}_{m} = \begin{cases} \begin{pmatrix} 1 \\ -im/|m| \end{pmatrix} & m \ne 0; \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & m = 0. \end{cases}$$

To the even index "e" at $\chi^{(e,o)}_{m}$ there corresponds a top line in parentheses (1 or 2), and to the odd one "o" the bottom in parentheses ($-im/|m|$ or 0) answers. Note that in expressions similar to (6) in the work [4], through an oversight of the authors, the lower indices m, m' are interchanged, and the signs are changed by opposite ones at "2" at $\tilde{\zeta}^{(e,o)}_{nm}$ and at $|m|$ in the factorials at $Q^{(e,o)}_{nm}$.

Adding together and subtracting the result of scalar products of (1) and (2) into basic vectorial functions $\hat{i}^{+}_{nm}$ and $\hat{i}^{-}_{nm}$, it is possible to derive the following infinite system of linear equations for Fourier's coefficients for the electromagnetic field (see Appendix):

$$\begin{cases} \Gamma^{\pm}(Z_n)\cdot\begin{pmatrix} i\tilde{B}^{(e,o)}_{nm} \\ \tilde{A}^{(e,o)}_{nm} \end{pmatrix} = \dfrac{N_{nm}}{\sqrt{2}}\langle U_{\pm nm}\rangle, & (8) \\ \Omega^{\pm}(Z_{n_2})\cdot\begin{pmatrix} i\tilde{b}^{(e,o)}_{n_2m_2} \\ \tilde{a}^{(e,o)}_{n_2m_2} \end{pmatrix} = \dfrac{N_{n_2m_2}}{\sqrt{2}} u_{n_2m_2}. & (9) \end{cases}$$

Here

$$\Omega^{+}(Z_n) = \left(\frac{1}{x}\frac{d}{dx}x + i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\right) Z_n(x), \quad (10)$$

$$\Omega^{-}(Z_n) = \left(1 - i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{x}\frac{d}{dx}x\right) Z_n(x) \quad (11)$$

are linear differential operators, and

$$\Gamma^{\pm}(Z_n) = \left(1 + \frac{k^2\sigma^2}{2}\frac{d^2}{dx^2}\right)\Omega^{\pm}(Z_n).$$

The upper sign "+" in equations (8), (9) is answered with the top line in parentheses $\left(i\tilde{B}^{(e,o)}_{nm}, i\tilde{b}^{(e,o)}_{nm}\right)$, and the lower sign "–" is answered with the bottom one $\left(\tilde{A}^{(e,o)}_{nm}, \tilde{a}^{(e,o)}_{nm}\right)$; $\left\{\tilde{A}^{(e,o)}_{nm}, \tilde{B}^{(e,o)}_{nm}\right\} = \left\{A^{(e,o)}_{n|m|}, B^{(e,o)}_{n|m|}\right\} Q^{(e,o)}_{nm}$ are Fourier coefficients renormalized. Through $u_{\pm n_2m_2}$, a linear combination of scalar products of vectorial functions ([5,6])

$$u_{\pm n_2m_2} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\left(\vec{u}\cdot\bar{\hat{i}}^{+}_{n_2m_2} \pm \vec{u}\cdot\bar{\hat{i}}^{-}_{n_2m_2}\right) \quad (12)$$

is designated. Here, horizontal line above basic vectorial functions $\bar{\hat{i}}^{\pm}_{n_2m_2}$ means complex conjugation. The variable $U_{\pm nm}$ is defined similarly. The calculations give the following expression for $u_{\pm n_2 m_2}$

$$u_{\pm n_2m_2} = \frac{1}{2\sqrt{2}N_{n_2m_2}}\times$$

$$\times\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{l=|n_1-n_3|}^{n_1+n_3}\tilde{\zeta}^{(e,o)}_{n_1m_1}\times$$

$$\times{}^{\pm}S^{m_3}_{n_1n_3l}\, C(n_1,n_3,l;m_1,m_3,m_1+m_3)\,\delta_{l,n_2}\delta_{m_2,m_1+m_3}, \quad (13)$$

where $C(n_1,n_3,l;m_1,m_3,m_1+m_3)$ are the Clebsch-Gordan coefficients and

$$ {}^{\pm}S^{m_3}_{n_1n_3l} = -{}^{\pm1}S^{m_3}_{n_1n_3l} - {}^{\pm2}S^{m_3}_{n_1n_3l} + {}^{\pm3}S^{m_3}_{n_1n_3l} + {}^{\pm4}S^{m_3}_{n_1n_3l} \quad (14)$$

is the sum of expansions of four summands $\vec{u}_i$ $(i=1,2,3,4)$ in the right part of (2b):

$$ {}^{\pm1}S^{m_3}_{n_1n_3l} = k\Big\{\tilde{A}^{(e,o)}_{n_3m_3} Z'_{n_3}(x)\left[-(-1)^{l-n_1-n_3} \pm 1\right] - i\tilde{B}^{(e,o)}_{n_3m_3}\left(\frac{1}{x}\left(xZ_{n_3}(x)\right)'\right)'\left[(-1)^{l-n_1-n_3} \pm 1\right]\Big\} \times C(n_1,n_3,l;0,1,1), \quad (15)$$

$$ {}^{\pm2}S^{m_3}_{n_1n_3l} = \frac{1}{ia}\sqrt{n_1(n_1+1)n_3(n_3+1)}\,\tilde{B}^{(e,o)}_{n_3m_3} \times \frac{1}{x} Z_{n_3}(x) C(n_1,n_3,l;1,0,1), \quad (16)$$

$$ {}^{\pm3}S^{m_3}_{n_1n_3l} = -ik\eta_0\sqrt{\frac{\varepsilon}{\mu}}\Big\{\tilde{A}^{(e,o)}_{n_3m_3}\left(\frac{1}{x}\left(xZ_{n_3}(x)\right)'\right)' \times \left[-(-1)^{l-n_1-n_3} \pm 1\right] + i\tilde{B}^{(e,o)}_{n_3m_3} Z'_{n_3}(x) \times \left[(-1)^{l-n_1-n_3} \pm 1\right]\Big\} C(n_1,n_3,l;0,1,1), \quad (17)$$

$$ {}^{\pm4}S^{m_3}_{n_1n_3l} = -\frac{i\eta_0}{a}\sqrt{\frac{\varepsilon}{\mu}}\sqrt{n_1(n_1+1)n_3(n_3+1)} \times \tilde{A}^{(e,o)}_{n_3m_3}\frac{1}{x}Z_{n_3}(x)\left[-(-1)^{l-n_1-n_3} \pm 1\right] C(n_1,n_3,l;1,0,1). \quad (18)$$

We shall represent $U_{\pm nm} = U^{(1)}_{\pm nm} + U^{(2)}_{\pm nm}$, where $U^{(1)}_{\pm\, nm} = U_{\pm5nm} - U_{\pm6nm}$ is received by means of transformations of the summand $\vec{U}_5 - \vec{U}_6$ in the right part of (1), and $U^{(2)}_{\pm nm}$ is a result of operations with the first four summands of the right part of (2a)

$$ U^{(2)}_{\pm nm} = +U_{\pm1nm} + U_{\pm2nm} - U_{\pm3nm} - U_{\pm4nm} = \frac{1}{2\sqrt{2}N_{nm}} \sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{l=|n_2-n_4|}^{n_2+n_4} \tilde{\zeta}^{(e,o)}_{n_4m_4} \times {}^{\pm}S^{m}_{n_4n_2l} C(n_4,n_2,l;m_4,m_2,m_4+m_2)\delta_{l,n}\delta_{m,m_4+m_2}. \quad (19)$$

The expression for ${}^{\pm}S^{m_2}_{n_4n_2l}$ can be derived from expression ${}^{\pm}S^{m_3}_{n_1n_3l}$ by replacement of $n_1 \to n_4$, $n_3 \to n_2$, $m_3 \to m_2$ and $\tilde{A}^{(e,o)}_{n_3m_3} \to \tilde{a}^{(e,o)}_{n_2m_2}$, $\tilde{B}^{(e,o)}_{n_3m_3} \to \tilde{b}^{(e,o)}_{n_2m_2}$.

If we introduce designations

$$ {}^{\pm A}S_{n_4n_2\tilde{l}} = -k\Big\{\frac{d}{dx}\Omega^-\left(Z_{n_2}\right)C(n_4,n_2,\tilde{l};0,1,1) - i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\sqrt{n_4(n_4+1)}\sqrt{n_2(n_2+1)}\frac{1}{x^2}Z_{n_2}(x) \times C(n_4,n_2,\tilde{l};1,0,1)\Big\}\left[(-1)^{\tilde{l}-n_4-n_2} \mp 1\right], \quad (20)$$

$$ {}^{\pm B}S_{n_4n_2\tilde{l}} = -k\Big\{\frac{d}{dx}\Omega^+\left(Z_{n_2}\right)C(n_4,n_2,\tilde{l};0,1,1) + \sqrt{n_4(n_4+1)n_2(n_2+1)}\frac{1}{x^2}Z_{n_2}(x) \times C(n_4,n_2,\tilde{l};1,0,1)\Big\}\left[(-1)^{\tilde{l}-n_4-n_2} \pm 1\right], \quad (21)$$

then

$$ {}^{\pm}S^{m_2}_{n_4n_2\tilde{l}} = \tilde{a}^{(e,o)}_{n_2m_2}\cdot{}^{\pm A}S_{n_4n_2\tilde{l}} + \tilde{b}^{(e,o)}_{n_2m_2}\cdot{}^{\pm B}S_{n_4n_2\tilde{l}}. \quad (22)$$

Similarly,

$$ {}^{\pm}S^{m_3}_{n_1n_3l} = \tilde{A}^{(e,o)}_{n_3m_3}\cdot{}^{\pm A}S_{n_1n_3l} + \tilde{B}^{(e,o)}_{n_3m_3}\cdot{}^{\pm B}S_{n_1n_3l}, \quad (23)$$

where ${}^{\pm A}S_{n_1n_3l}$ is derived from ${}^{\pm A}S_{n_4n_2\tilde{l}}$, and ${}^{\pm B}S_{n_1n_3l}$ from ${}^{\pm B}S_{n_4n_2\tilde{l}}$ by replacement $n_4 \to n_1$, $n_2 \to n_3$, $\tilde{\ell} \to \ell$. Solving the system (9) for the variables $\tilde{a}^{(e,o)}_{n_2m_2}$ $\tilde{b}^{(e,o)}_{n_2m_2}$ and substituting them into $\langle U^{(2)}_{\pm nm}\rangle$, we shall receive

$$ \langle U^{(2)}_{\pm nm}\rangle = \frac{1}{8\sqrt{2}N_{nm}} \sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}\sum_{\tilde{l}=|n_2-n_4|}^{n_2+n_4}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} \sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{l=|n_1-n_3|}^{n_1+n_3} \langle \tilde{\zeta}^{(e,o)}_{n_4m_4}\cdot\tilde{\zeta}^{(e,o)}_{n_1m_1}\rangle \Big\{{}^{\pm A}S_{n_4m_2l}\left[\Omega^-\left(Z_{n_2}\right)\right]^{-1} \times $$

$$\times\left[{}^{-A}S_{n_1n_3l}\widetilde{A}^{(e,o)}_{n_3m_3}+{}^{-B}S_{n_1n_3l}\ \widetilde{B}^{(e,o)}_{n_3m_3}\right]-i\ {}^{\pm B}S_{n_4n_2l}\left[\Omega^{+}\left(Z_{n_2}\right)\right]^{-1}\times$$

$$\times\left[{}^{+A}S_{n_1n_3l}\widetilde{A}^{(e,o)}_{n_3m_3}+{}^{+B}S_{n_1n_3l}\ \widetilde{B}^{(e,o)}_{n_3m_3}\right]\Big\}\ \delta_{\widetilde{l},n}\,\delta_{m,m_4+m_2}\delta_{l,n_2}\delta_{m_2,m_1+m_3}\ . \qquad (24)$$

The product of four Kronecker symbols in this formula is a result of orthonormalization of the chosen system of basic vectorial functions (6).

For a homogeneous and isotropic field of random irregularities $\zeta(\varphi,\theta)$, correlator of spectral amplitudes according to Obukhov's theorem [9,10] is related to a power spectrum of a correlation function $K(\Theta)$ by a ratio that is similar to Winer-Khinchin's theorem:

$$\left\langle\widetilde{\zeta}^{(e,o)}_{nm}\widetilde{\zeta}^{(e,o)}_{n'm'}\right\rangle=(-1)^m(4n+2)\,\delta_{n,n'}\delta_{-m,m'}\times$$

$$\times\int\limits_0^{\pi}K(\Theta)P^n_{0,0}(\cos\Theta)\sin\Theta d\Theta\,, \qquad (25)$$

where $K(\Theta)=\left\langle\zeta(\varphi_1,\theta_1)\cdot\zeta(\varphi_2,\theta_2)\right\rangle$, is the correlation function, and $\Theta-$ is an angular distance between points (φ_1,θ_1) and (φ_2,θ_2).

In the case of small correlation radii $\ell_{cor}<<a$, it is possible to continue analytically a correlation function $B(\rho)$ determined in a tangent plane, on a sphere, assuming $\theta\approx\rho/a$ and $B(\rho)\approx K(\rho/a)$. If we take into account an asymptotic constraint ([11], Chapter 2)

$$P^n_{0,0}(\cos\theta)=P_n(\cos\theta)=J_0\left((2n+1)\sin\frac{1}{2}\theta\right)+$$

$$+O\left(\sin^2\frac{1}{2}\theta\right)\approx J_0(n\theta)\quad(\theta<<1), \qquad (26)$$

then (25) takes the form

$$\left\langle\widetilde{\zeta}^{(e,o)}_{nm}\cdot\widetilde{\zeta}^{(e,o)}_{n'm'}\right\rangle\approx4\pi(\sigma/a)^2(-1)^m\times$$

$$\times(2n+1)\,\delta_{n,n'}\delta_{-m,m'}\,\widetilde{W}(n/a), \qquad (27)$$

where

$$\widetilde{W}(\chi)=\left(2\pi\sigma^2\right)^{-1}\int\limits_0^{\infty}B(\rho)J_0(\chi\rho)\rho d\rho \qquad (28)$$

is an isotropic spectrum of irregularities in the tangent plane; $\chi=n/a$. In case of inapplicability of the tangent plane approximation, everywhere below, by $\widetilde{W}(\chi)$ it is necessary to understand an expression following from initial (25)

$$\widetilde{W}(\chi)=\left(2\pi\sigma^2\right)^{-1}a^2\times$$

$$\times\int\limits_0^{\pi}K(\Theta)P^n_{0,0}(\cos\Theta)\sin\Theta\,d\Theta. \qquad (29)$$

Thus, within the ten-multiple sum sign, after substitution of (25), there will be a product of 6 Kronecker symbols. The properties of orthogonality of Clebsch-Gordan's factors (see. Appendix) add into the product, as cofactors, two more Kronecker symbols. In total, a product of 8 Kronecker symbols is turned out.

Choosing correctly the contribution of the summation areas on each variable, it is possible to perform summation with respect to the 8 sums (see Appendix), and, as the result, we obtain

$$U^{(2)}_{\pm n}=\frac{N_{nm}}{\sqrt{2}}\left\langle U^{(2)}_{\pm nm}\right\rangle\Big/\begin{pmatrix}i\widetilde{B}^{(e,o)}_{nm}\\ \widetilde{A}^{(e,o)}_{nm}\end{pmatrix}=$$

$$=O^{(1)}_{\pm n}\hat{Z}_n{}'(x)+O_{\pm n}\hat{Z}_n(x), \qquad (30)$$

where $\hat{Z}_n(x)=xZ_n(x)$ are Riccati-Bessel functions, and

$$O^{(1)}_{\pm n}=\frac{\pi}{2}\left(\frac{k\sigma}{a}\right)^2\times$$

$$\times\sum_{n_4=0}^{\infty}\sum_{l=|n-n_4|}^{n+n_4}(2n_4+1)\widetilde{W}(n_4/a)G^{\pm}_{n,n_4,l}\cdot L^{\pm(1)}_{nn_4l}, \qquad (31)$$

$$O_{\pm n}=\frac{\pi}{2}\left(\frac{k\sigma}{a}\right)^2\times$$

$$\times\sum_{n_4=0}^{\infty}\sum_{l=|n-n_4|}^{n+n_4}(2n_4+1)\widetilde{W}(n_4/a)G^{\pm}_{n,n_4,l}\cdot L^{\pm}_{nn_4l}. \qquad (32)$$

Thereby, the designations are used

$$G^{\pm}_{n,n_4,l} = C(n,n_4,l;1,0,1)\left[\frac{1\mp(-1^{n-n_4-l})}{\Omega^-(Z_l)}\left(\frac{d}{dx}\Omega^-(Z_l)+i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\frac{l(l+1)}{x^2}Z_l\right)+\right.$$

$$\left.+\frac{1+(-1)^{n-n_4-l}}{\Omega^+(Z_l)}\left(\frac{d}{dx}\Omega^+(Z_l)-\frac{l(l+1)}{x^2}Z_l\right)\right]\pm\begin{pmatrix}1\\ i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\end{pmatrix}\times$$

$$\times C(n,n_4,l;0,0,0)\sqrt{l(l+1)n(n+1)}\frac{1}{x^2}Z_l\frac{1+(-1)^{n-n_4-l}}{\Omega^{\pm}(Z_l)}, \qquad (33)$$

$$L^{+(1)}_{nn_4l} = C(n,n_4,l;1,0,1)\frac{1}{x}\left(i\eta_0\sqrt{\frac{\varepsilon}{\mu}}-\frac{1}{x}\right), \qquad (34)$$

$$L^{-(1)}_{nn_4l} = C(n,n_4,l;1,0,1)\frac{1}{x}\left(1+i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{x}\right), \qquad (35)$$

$$L^{+}_{nn_4l} = -C(n,n_4,l;1,0,1)\frac{1}{x}\left(1+i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{x}\right)+$$

$$+C(n,n_4,l;0,0,0)\sqrt{l(l+1)n(n+1)}\frac{1}{x^3}, \qquad (36)$$

$$L^{-}_{nn_4l} = C(n,n_4,l;1,0,1)\frac{1}{x}\left(i\eta_0\sqrt{\frac{\varepsilon}{\mu}}-\frac{1}{x^2}\right)-$$

$$-C(n,n_4,l;0,0,0)\cdot\frac{i\eta_0}{x^3}\sqrt{\frac{\varepsilon}{\mu}}\sqrt{l(l+1)n(n+1)}. \qquad (37)$$

Similarly,

$$U^{(1)}_{\pm n} = \frac{N_{nm}}{\sqrt{2}}\left\langle U^{(1)}_{\pm nm}\right\rangle\Big/\begin{pmatrix}i\tilde{B}^{(e,o)}_{nm}\\ \tilde{A}^{(e,o)}_{nm}\end{pmatrix} = U_{\pm 5n}-U_{\pm 6n}, \qquad (38)$$

thereby,

$$U_{\pm 5n} = O_{\pm 5n}\begin{pmatrix}\hat{Z}_n(x)\\ \hat{Z}_n'(x)\end{pmatrix},$$

$$U_{\pm 6n} = O_{\pm 6n}\begin{pmatrix}\hat{Z}_n(x)\\ \hat{Z}_n'(x)\end{pmatrix}. \qquad (39)$$

The expressions $O_{\pm 5n}, O_{\pm 6n}$ are adduced in Appendix. Substitution of the expressions for $U^{(1)}_{\pm n}$ and $U^{(2)}_{\pm n}$ in the right part of (8), after separation of fields into incident and scattered one, makes possible to solve the system of equations for Fourrier coefficients of the scattered field

$$\tilde{A}^{s(e,o)}_{nm} = -\frac{(\Gamma_0^- - xO_{-n})\hat{\psi}_n(x)+\left[\Gamma_1^- - x\left(O^{(1)}_{-n}+O_{-5n}+O_{-6n}\right)\right]\hat{\psi}_n'(x)}{(\Gamma_0^- - xO_{-n})\hat{\zeta}^{(1)}_n(x)+\left[\Gamma_1^- - x\left(O^{(1)}_{-n}+O_{-5n}+O_{-6n}\right)\right]\hat{\zeta}^{(1)\prime}_n(x)}\tilde{A}^{0(e,o)}_{nm}, \qquad (40)$$

$$\tilde{B}^{s(e,o)}_{nm} = -\frac{(\Gamma_1^+ - xO^{(1)}_{+n})\hat{\psi}_n'(x)+\left[\Gamma_0^+ - x\left(O_{+n}+O_{+5n}+O_{+6n}\right)\right]\hat{\psi}_n(x)}{(\Gamma_1^+ - xO^{(1)}_{+n})\hat{\zeta}^{(1)\prime}_n(x)+\left[\Gamma_0^- - x\left(O_{+n}+O_{+5n}+O_{+6n}\right)\right]\hat{\zeta}^{(1)}_n(x)}\tilde{B}^{0(e,o)}_{nm}, \qquad (41)$$

with

$\hat{\psi}_n(x)=\sqrt{\pi x/2}\cdot J_{n+\frac{1}{2}}(x)$, $\hat{\zeta}^{(1,2)}_n(x) = \sqrt{\pi x/2}\cdot H^{(1,2)}_{n+\frac{1}{2}}(x)$

being Riccatti-Bessel spherical functions, the upper indices s, o corresponding to Fourrier coefficients of scattered and initial field, respectively. It is necessary to take into account that

$\hat{\psi}_n(x)=\frac{1}{2}\left[\hat{\zeta}^{(1)}_n(x)+\hat{\zeta}^{(2)}_n(x)\right]$, as well as

$$\Gamma_1^+ = 1+\Delta\Gamma_1^+ = 1+\tfrac{1}{2}k^2\sigma^2\left[\frac{n(n+1)+2}{x^2}-1-i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\frac{2}{x}\right], \qquad (42)$$

$$\Gamma_1^- = -i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + \Delta\Gamma_1^- = -i\eta_0\sqrt{\frac{\varepsilon}{\mu}} - \frac{1}{2}k^2\sigma^2 \times \left[\frac{2}{x} + i\eta_0\left(\frac{n(n+1)+2}{x^2} - 1\right)\right], \quad (43)$$

$$\Gamma_0^+ = i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + \Delta\Gamma_0^+ = i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + \frac{1}{2}k^2\sigma^2 \times \left[\frac{2}{x}\left(1 - \frac{2n(n+1)}{x^2}\right) + i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{n(n+1)+2}{x^2} - 1\right)\right], \quad (44)$$

$$\Gamma_0^- = 1 + \Delta\Gamma_0^- = 1 + \frac{1}{2}k^2\sigma^2 \times \left[\frac{n(n+1)+2}{x^2} - 1 - i\eta_0\sqrt{\frac{\varepsilon}{\mu}}\frac{2}{x}\left(1 - \frac{2n(n+1)}{x^2}\right)\right]. \quad (45)$$

Let us divide numerator and denominator in (40) by $\Gamma_1^+ - xO_{+n}^{(1)}$, and those in (41) by $\Gamma_0^- - xO_{-n}$, then expand the result of division in a power series of small additives $\Delta\Gamma_1^+$, $\Delta\Gamma_0^-$. We yield

$$B_{nm}^{s(e,o)} = -\frac{1}{2}\left[1 - R_n^E \frac{\hat{\zeta}_n^{(2)}(x)}{\hat{\zeta}_n^{(1)}(x)}\right] \cdot B_{nm}^{0(e,o)}, \quad (46)$$

$$A_{nm}^{s(e,o)} = -\frac{1}{2}\left[1 - R_n^M \frac{\hat{\zeta}_n^{(2)}(x)}{\hat{\zeta}_n^{(1)}(x)}\right] \cdot A_{nm}^{0(e,o)}, \quad (47)$$

where R_n^E, R_n^M have the meaning of reflection coefficients of spherical waves of the multipoles of order of n in expansion of a coherent field into series, and they look as follows

$$R_n^E = -\frac{\ln'\hat{\zeta}_n^{(2)}(x) + i\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}}}{\ln'\hat{\zeta}_n^{(1)}(x) + i\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}}}, \quad (48)$$

$$R_n^M = -\frac{1 + i\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}}\ln'\hat{\zeta}_n^{(2)}(x)}{1 + i\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}}\ln'\hat{\zeta}_n^{(1)}(x)}, \quad (49)$$

where the effective impedances are determined by expressions

$$i\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} = i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + i\Delta\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} = i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + i\sqrt{\frac{\varepsilon}{\mu}}\left(\Delta_0\eta_{eff}^E + \Delta_1\eta_{eff}^E + \Delta_2\eta_{eff}^E\right), \quad (50)$$

$$i\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} = i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + i\Delta\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} = i\eta_0\sqrt{\frac{\varepsilon}{\mu}} + i\sqrt{\frac{\varepsilon}{\mu}}\left(\Delta_0\eta_{eff}^M + \Delta_1\eta_{eff}^M + \Delta_2\eta_{eff}^M\right), \quad (51)$$

and

$$i\Delta_0\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} = \frac{1}{2}k^2\sigma^2\frac{2}{x}\left(1 - \frac{2n(n+1)}{x^2}\right), \qquad i\Delta_0\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} = \frac{1}{2}k^2\sigma^2\frac{2}{x}, \quad (52)$$

$$i\Delta_1\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} = -x\left(O_{+5n} + O_{+6n}\right), \qquad i\Delta_1\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} = x\left(O_{-5n} + O_{-6n}\right), \quad (53)$$

$$i\Delta_2\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} = i\eta_0\sqrt{\frac{\varepsilon}{\mu}}xO_{+n}^{(1)} - xO_{+n}, \qquad i\Delta_2\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} = i\eta_0\sqrt{\frac{\varepsilon}{\mu}}xO_{-n} + xO_{-n}^{(1)}. \quad (54)$$

Conclusion

Thus, at unspecified finite values $x = ka$, the relationships (50), (51), (52), (53), and (54) solve the problem on an effective impedance of a mean (coherent) field scattered by a statistically rough sphere, and, hence, that of the mean field as well.

An important for applications case of a large sphere $(ka >> 1)$ is treated analytically in details in Part II of the present work published in this issue.

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Эффективный импеданс статистически неровной сферы: I. Общий случай

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В приближении Бурре проанализировано среднее электромагнитное поле, рассеянное статистически неровной сферой с малым поверхностным импедансом η_0. Вычислены коэффициенты отражения среднего поля, которые выражены через эффективные импедансы сферических мультипольных волн. В отличие от предшествующего исследования авторов, учтены не только возмущения, пропорциональные дисперсии высот неровностей σ^2, но и $\sim \eta_0\sigma^2$, что необходимо при недостаточно малых η_0.

Ефективний імпеданс статистично нерівної сфери: I. Загальний випадок

А.С. Брюховецький, Л.О. Пазинін

В наближенні Бурре проаналізовано середнє електромагнітне поле, розсіяне статистично нерівною сферою з малим поверхневим імпедансом η_0. Обчислено коефіцієнти відбиття середнього поля, що виражені через ефективні імпеданси сферичних мультипольних хвиль. На відміну від попереднього дослідження авторів враховано не тільки збурення, пропорційні дисперсії висот нерівностей σ^2, але й $\sim \eta_0\sigma^2$, що необхідно при недостатньо малих η_0.