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Effective Impedance of a Statistically Rough Sphere: II. A Case of the Large Sphere

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The expressions for spheric wave impedance obtained in Part I are simplified, using asymptotic expansions for high-frequency scattering on small-scaled irregularities. Transitions are studied both to the limit of impedance of spherical waves without taking into account perturbations $\sim \eta_0\sigma^2$ (η_0 is non-perturbed impedance, σ^2 is dispersion of heights of the irregularities) and to the limit of impedance of plane waves scattered by statistically rough plane.

Introduction

The relations for an effective impedance of the mean (coherent) electromagnetic field scattered by a statistically rough sphere were derived in Part I (see preceding paper) without any restrictions on parameter $x = ka$ value. Here we consider a case of large sphere $(x >> 1, n \sim x)$ having small-scale irregularities $(\chi = n_4/a \sim k, n_4 >> 1)$.

Effective impedance: Asymptotic representation for $\chi a, ka >> 1$

Let us take asymptotic representations of the values entering the expressions (50), (51) and proceed from summation with respect to n_4, l to integration over them. Formulae numeration in this Part is an extention to that of the Part I. Thereby, the following relation is necessary to be implied:

a) if $\alpha = n - n_4 - l$ is even, then

$$1+(-1)^\alpha = 2,\ 1-(-1)^\alpha = 0 \text{ and}$$

$$C(n,n_4,l;1,0,1) \approx \frac{l^2+n^2-n_4^2}{2nl} C(n,n_4,l;0,0,0),$$

$$C(n,n_4,l;1,0,1) \approx (-1)^{\alpha/2} \frac{2\sqrt{l/\pi}}{\sqrt[4]{4n^2n_4^2-\left(l^2-n^2-n_4^2\right)^2}};$$

b) if $\alpha = n - n_4 - l$ is odd, then

$$1+(-1)^\alpha = 0,\ 1-(-1)^\alpha = 2 \text{ and}$$

$$C(n,n_4,l;0,0,0,) \equiv 0,$$

$$C(n,n_4,l;1,0,1) \approx \frac{\sqrt{\left[(n+n_4)^2-l^2\right]\left[l^2-(n-n_4)^2\right]}}{2nl} \times$$

$$\times C(n-1,n_4-1,l-1;0,0,0),$$

$$C(n-1,n_4-1,l-1;0,0,0) \approx (-1)^{\frac{\alpha+1}{2}} \times$$

$$\times \frac{2\sqrt{l/\pi}}{\sqrt[4]{4n^2n_4^2-\left(l^2-n^2-n_4^2\right)^2}}.$$

Then

$$(x \to \infty):\ \Delta_0\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} \to 0,\ \Delta_0\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} \to 0, \text{ and}$$

(see Appendix)

$$\Delta_1\eta_{eff}^E\sqrt{\frac{\varepsilon}{\mu}} = \eta_0\sqrt{\frac{\varepsilon}{\mu}}\sigma^2 \times$$

$$\times \int_0^\infty d\chi \int_0^{2\pi} d\varphi \chi \tilde{W}(\chi) \frac{\chi_x^2-\chi_y^2}{2}, \tag{55}$$

$$\Delta_1\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}} = \eta_0\sqrt{\frac{\varepsilon}{\mu}}\sigma^2 \times$$

$$\times \int_0^\infty d\chi \int_0^{2\pi} d\varphi \chi \tilde{W}(\chi) \frac{\chi_y^2-\chi_x^2}{2}. \tag{56}$$

For a sphere in vacuum $\left(\sqrt{\varepsilon/\mu} = 1\right)$, the formula (55) transforms into an expression for $\Delta_1\eta_{xx}$, and (56) does into an expression for $\Delta_1\eta_{yy}$, in a problem of scattering by a rough plane [1]. The difference is only that [1] contains a spectrum $S(\vec{\chi})$, normalized on σ^2, which, for isotropic case, is equal

$S(\vec{\chi})=S(\chi)$, and is connected with $\tilde{W}(\chi)$ by means of equation $S(\chi)=\sigma^2\tilde{W}(\chi)$. We shall introduce designations $k_\perp = n/a, k_z=\sqrt{k^2-k_\perp^2}$, $\eta_0'=\eta_0\sqrt{\varepsilon/\mu}$ and $\nu = l/a$, $\ln'\zeta_l^{(1)}(x)=i(\chi_z/k)$ (for Debay's asymptotic $i(\chi_z/k)=i\sqrt{1-(l/x)^2}$). Then the transition from summation to integration gives rise to the result

$$\Delta_2\eta_{eff}^E\cdot\sqrt{\frac{\varepsilon}{\mu}}=(k\sigma)^2\int_0^\infty \chi\tilde{W}(\chi)d\chi\times$$
$$\times\int_0^{2\pi}d\varphi\left\{\frac{1}{k^3(\chi_z+k\eta_0')}\times\right.$$
$$\times\left[k_z^2\chi_z^2+k^2\chi^2+\eta_0'k\chi_z(k_z^2-k_\perp\chi\cos\varphi)-\right.$$
$$\left.-\eta_0'^2k^2(k_z^2-k_\perp\chi\cos\varphi)-\eta_0'^3k^3\chi_z\right]-$$
$$\left.-(1-\eta_0'^2)^2\frac{\chi^2\sin^2\varphi}{(\chi_z+k\eta_0')(k+\chi_z\eta_0')}\right\}, \qquad (57)$$

$$\Delta_2\eta_{eff}^M\sqrt{\frac{\varepsilon}{\mu}}=(k\sigma)^2\int_0^\infty\chi\,\tilde{W}(\chi)\,d\chi\int_0^{2\pi}d\varphi\left\{\frac{1}{k+\chi_z\eta_0'}\times\right.$$
$$\times\left[\chi_z+\eta_0'k\left(1-\frac{k_\perp^2+k_\perp\chi\cos\varphi}{k^2}\right)-\eta_0'^2\left(\chi_z+\eta_0'k-\right.\right.$$
$$\left.\left.-\frac{(k_\perp^2+k_\perp\chi\cos\varphi)(\chi_z+\eta_0'2k)}{k^2}\right)+\frac{\eta_0'k\nu^2k_\perp^2}{k^4}\right]+$$
$$\left.+(1-\eta_0'^2)^2\frac{\chi^2\sin^2\varphi}{(\chi_z+k\eta_0')(k+\chi_z\eta_0')}\right\}. \qquad (58)$$

Obviously, by putting $\eta_0'=0$ in numerators of (57) and (58), we come to the formulas (18) and (19) of [2] for the case of $\sqrt{\varepsilon/\mu}=1$.

This kind of procedure corresponds to preservation of only first two members in the right hand parts of (2a) and (2b), as it was just used in the work [2]. To compare (57) and (58) with results for a rough plane [1], we shall bring (57) and (58) into a form

$$i\Delta_2\eta_{eff}^{E,M}\sqrt{\frac{\varepsilon}{\mu}}=(k\sigma)^2\times$$
$$\times\int_0^\infty\chi\,\tilde{W}(\chi)d\chi\int_0^{2\pi}d\varphi\,\frac{\sum_{j=0}^4(\eta_0')^jC_j^{e,m}}{(\chi_z+k\eta_0')(k+\chi_z\eta_0')}. \qquad (59)$$

Thereby,

$$C_0^e=k(k_z^2\chi_z^2+k^2\chi_x^2),$$
$$\chi_x=\chi\cos\varphi,\quad \chi_y=\chi\sin\varphi,$$
$$C_1^e=\chi_z\{k_z^2\chi_z^2+k^2\chi^2+\tfrac{1}{2}k^2(k_z^2+\chi_z^2+\chi^2)\},$$
$$C_2^e=k(\chi_z^2-k^2)(k_z^2-k_\perp\chi_x)+2k^3\chi_y^2, \qquad (60)$$
$$C_3^e=-\chi_zk^2(k^2+k_z^2-k_\perp\chi_x),$$
$$C_4^e=-k^3(\chi_z^2+\chi_y^2),$$

$$C_0^m=k^3(\chi_z^2+\chi_y^2),$$
$$C_1^m=k^2\chi_z[2k^2-k_\perp(k_\perp+\chi_x)],$$
$$C_2^m=k^3\{k^{-2}(k^2-\chi_z^2)[k^2-k_\perp(k_\perp+\chi_x)]-2\chi_y^2\},$$
$$C_3^m=k\{-k\chi_z[2k^2-3k_\perp(k_\perp+\chi_x)]+\nu^2k_\perp^2\}, \qquad (61)$$
$$C_4^m=k^3\{-k^{-2}[k^4-2k_\perp(k_\perp+\chi_x)k^2+\nu^2k_\perp^2]+\chi_y^2\}.$$
$$\nu^2=k_\perp^2+\chi^2+2k_\perp\chi\cos\varphi.$$

For comparison with components of the effective impedance tensor $\Delta_2\eta_{\alpha\beta}$ (p.1403 of [1]):

$$\Delta_2\eta_{\alpha\beta}=k^{-1}\times$$
$$\times\iint d^2\kappa\cdot S(\vec{\kappa}-\vec{k}_\perp)\frac{\sum_{j=0}^4\eta_0^jC_{\alpha\beta}^{(j)}}{(\kappa_z+k\eta_0)(k+\kappa_z\eta_0)}, \qquad (62)$$
$$\kappa_z=\sqrt{k^2-\kappa^2},\quad (\alpha,\beta)=(x,y)$$

it is necessary to take into account the difference in designations $(k_0\to k, \vec{k}\to\vec{k}_\perp)$ and to convert to a new integration variable $\vec{\chi}=\vec{\kappa}-\vec{k}_\perp$, thereby $\kappa_z=\sqrt{k^2-\kappa^2}=\sqrt{k^2-\chi^2-k_\perp^2-2\vec{k}_\perp\vec{\chi}}=\chi_z$.

Note that the expression $C_{\alpha\beta}^{(4)}$ (page 1403 of [1]) contains a misprint: the last summand in a square bracket should be $-k_\alpha k_\beta$, instead of $+k_\alpha k_\beta$.

Let us illustrate the agreement between the spherical problem asymptotic and the solution [1] by way of example $C_{xx}^{(0)}, C_{xx}^{(1)}, C_{xx}^{(4)}$, where volume of transformations is a minimal. Choose a coordinate system with an Ox axis being along the direction of $\vec{k}_\perp$, i.e. $\vec{k}_\perp=(k_\perp,0)$. Then

$$C_{xx}^{(0)} = k\kappa_z^2\left(k^2 - k_\perp^2\right) + k^3\left(\kappa_x - k_\perp\right)^2 =$$
$$= k\chi_z^2 k_z^2 + k^3\chi_x^2 = k\left(\chi_z^2 k_z^2 + k^2\chi_x^2\right),$$

where $\chi_z^2 = k^2 - k_\perp^2 - \chi^2 - 2k_\perp\chi\cos\varphi$ coincides with χ_z of (57) and (58), and k_z has the same sense, if $k_\perp = n/a = (ka\sin\beta)/a = k\sin\beta$, where β is a glancing angle at a mirror point, which is received from Watson method for a field asymptotics in the illuminated zone [1]. Thus, in the illuminated zone $C_{xx}^{(0)} = C_0^e$,

$$C_{xx}^{(1)} = \chi_z\left\{k^2\left[2k^2 - k_\perp\left(\vec{\chi}+\vec{k}_\perp\right) + \chi_y^2 - \left(\chi_x + k_\perp\right)^2\right] - \chi_z^2 k_\perp^2\right\} =$$
$$= \chi_z\left\{k^2\left(k^2 + k_z^2\right) - k^2 k_\perp\chi_x - \chi_z^2 k_\perp^2 + k\chi_x^2 - \right.$$
$$\left. - k\left(\chi_x^2 + 2\vec{k}_\perp\vec{\chi} + k_\perp^2\right)\right\} =$$
$$= \chi_z^2\left\{k^2\left(k^2 + k_z^2 - k_\perp^2\right) - \chi_z^2 k_\perp^2 - 3k^2\vec{k}_\perp\vec{\chi}\right\}.$$

As $\vec{k}_\perp\vec{\chi} = \frac{1}{2}\left(k_z^2 - \chi_z^2 - \chi^2\right)$, which follows from definition of χ_z^2, then

$$C_{xx}^{(1)} = \chi_z\left\{k_z^2\chi_z^2 + k^2\chi^2 + \tfrac{1}{2}k^2\left(k_z^2 + \chi_z^2 + \chi^2\right)\right\},$$

which completely coincides with C_1^e in the illuminated zone.

$$C_{xx}^{(4)} = k\left(k^2 - \vec{k}_\perp\vec{\kappa}\right)\left[-k^2 + \vec{k}_\perp\vec{\kappa} + \kappa_x^2 - \right.$$
$$\left. - 2\vec{k}_\perp\vec{\kappa} + k_\perp^2 - k_\perp^2\right] - k\left(k_\perp^2 - \vec{k}_\perp\vec{\kappa}\right)\chi_x^2 =$$
$$= k\left(k^2 - \vec{k}_\perp\vec{\kappa}\right)\times$$
$$\times\left(-k^2 - \vec{k}_\perp\vec{\kappa}\right) + k\left(k^2 - \vec{k}_\perp\vec{\kappa} - k_\perp^2 + \vec{k}_\perp\vec{\kappa}\right) =$$
$$= k\left(k^2 - \vec{k}_\perp\vec{\kappa}\right)\left(-k^2 - \vec{k}_\perp\vec{\kappa}\right) + k\,k_z^2\kappa_x =$$
$$= -k\left(k^4 - k_\perp^2\kappa_x^2\right) + k\,k_z^2\kappa_x^2 =$$
$$= k\left(-k^4 + k_\perp^2\kappa_x^2 + k_z^2\kappa_x^2\right) = k\left(-k^4 + k^2\kappa_x^2\right) =$$
$$= -k^3\left(k^2 - \kappa_x^2\right) = -k^3\left(k^2 - \kappa^2 + \kappa_y^2\right) =$$
$$= -k^3\left(\kappa_z^2 + \kappa_y^2\right) = -k^3\left(\chi_z^2 + \chi_y^2\right),$$

which, in the illuminated zone, coincides with C_4^e. The reader may convince of an agreement of other coefficients $C_{\alpha\beta}^{(n)}$ with each other, by performing the transformations similar to ones presented above.

In the shadow zone, the Watson method results in representation of the field as its expansion in terms of waves with complex value of n_s, and, hence, with complex both $k_\perp = n_s/a$ and $k_z = \sqrt{k^2 - k_\perp^2}$ that enter the coefficients $C_j^{e,m}$.

It is known ([3], Chapter 1) that $n_s \approx ka + (ka)^{1/3}\cdot O_s$, where $s = 0,1,2...$, and $|O_s|$, for the first five values of s, lies in the region of values $1\div 10$. Therefore, $n_s/a \approx k\left[1 + (ka)^{-2/3}O_s\right]$ and, correspondingly,

$$k_z^2 \approx k^2\left\{1 - 1 - 2(ka)^{-2/3}O_s\right\} \approx k^2\left\{-2(ka)^{-2/3}O_s\right\}.$$

From that, k_z is complex and is equal to $k_z^2/k^2 \ll 1$, while for plane waves glancing along the plane, $k_z^2 \equiv 0$. This difference does contain, in high-frequency case, the influence of curvature of sphere on additional decaying of a coherent field, which rises of wave being scattered on roughnesses of the sphere.

As a numerical example, we shall put results of calculations of $\Delta_2\eta_{eff}^E$ according to the formula (59) of the present work and of those in accordance with the formula (18) of [2] for the disturbed water surface (See Table). Let us accept for air $\sqrt{\varepsilon/\mu} = 1$. The dielectric constant of the water is equal to 80, the conductivity of it is $\sigma_0 = 4$ S/m (sea water). Spatial spectrum of irregularities

$$S(\chi) = \begin{cases} 5{,}10^{-3}/\left(2\pi\chi^4\right), & \chi \geq g/V^2, \\ 0, & \chi < g/V^2, \end{cases}$$

corresponds to a "half-isotropic" amplitude Phillips'es spectrum of the rough-developed sea at wind speed V assumed, in the calculations, being equal $V = 5\div 15$ m/s, and $g = 9{,}8$ m/s^2 is an acceleration of gravity.

The small difference of computational values at $\sigma_0 = 4$ S/m ($|\eta_0| \sim 8\cdot 10^{-3} \div 2\cdot 10^{-2}$) and rather sizeable one $(\sim 3\div 15\%)$ at $\sigma_0 = 4\cdot 10^{-3}$ S/m $\left(|\eta_0| \sim 10^{-1}\right)$ is explained by the fact that the representation $\Delta_2\eta_{eff}^E$ from [2], as being similar to (59), contains in its numerator only terms of $\tilde{C}_0^e + \eta_0\tilde{C}_1^e$, and thereby $\tilde{C}_0^e \equiv C_0^e$, and $\tilde{C}_1^e = \chi_z k^2\left(k_z^2 + \chi^2\right)$. For this reason, the difference in numerators of (59) and (18) of [2] amounts to

$\frac{1}{2}\eta_0 \chi_z k^2(k_z^2+\chi_z^2+\chi^2) \sim \eta_0 k^2 \chi^3$ at $\chi >> k$. If the spectrum of irregularities $S(\chi)$ is concentrated in the wave number region of $\chi \geq k/|\eta_0|$, the difference (59) from (18) of [4] can be much more essential.

Conclusions

The result of the researches carried out is the construction of a theory of small disturbances for scattering by statistically rough sphere, to within the terms of expansion of $\sim \eta_0 \sigma^2$, which is necessary at insufficiently small η_0, as well as for small-scale irregularities $(\chi >> k)$. Theoretical laws obtained in the tangent plane approximation $(\chi a >> 1)$ for an effective impedance in high-frequency case ($ka >> 1$) can be a basis for both theoretical evaluations of influence of the roughnesses on electromagnetic wave attenuation in the zone of shadow and testing of the heuristic methods of such evaluations. In limiting-cases, the transition to the results already known, for both the statistically rough sphere and the plane, is satisfied. For the general case of arbitrary ratio between the wavelength, the radius of the sphere and

Table

ε=80		j=0÷4, formula (59)		[2], formula (18)	
V,m/s	F,MHz	$\Delta_2\eta_{eff}^E(n)$		$\Delta_2\eta_{eff}^E(n)$	
$\sigma_0 = 4$		Re	Im	Re	Im
5	5	4,084E-04	-7,480E-03	4,092E-04	-7,475E-03
	10	1,923E-03	-1,760E-02	1,926E-03	-1,761E-02
	20	1,343E-02	-2,613E-02	1,346E-02	-2,614E-02
	30	2,016E-02	-3,066E-02	2,018E-02	-3,065E-02
10	5	1,307E-02	-2,564E-02	1,309E-02	-2,562E-02
	10	2,469E-02	-3,362E-02	2,471E-02	-3,361E-02
	20	3,861E-02	-4,484E-02	3,868E-02	-4,487E-02
	30	4,863E-02	-5,395E-02	4,877E-02	-5,397E-02
15	5	2,687E-02	-3,438E-02	2,688E-02	-3,440E-02
	10	4,098E-02	-4,694E-02	4,101E-02	-4,695E-02
	20	6,004E-02	-6,507E-02	6,020E-02	-6,438E-02
	30	7,440E-02	-7,724E-02	7,469E-02	-7,746E-02
σ_0=0,004					
5	5	,2544E-03	-,1021E-02	,2614E-03	-,1014E-02
	10	,6159E-03	-,2008E-02	,6301E-03	-,1990E-02
	20	,1812E-02	-,2785E-02	,1845E-02	-,2732E-02
	30	,2505E-02	-,3221E-02	,2549E-02	-,3131E-02
10	5	,1797E-02	-,2776E-02	,1826E-02	-,2727E-02
	10	,3003E-02	-,3594E-02	,3066E-02	-,3472E-02
	20	,4377E-02	-,4774E-02	,4537E-02	-,4493E-02
	30	,5289E-02	-,5776E-02	,5587E-02	-,5310E-02
15	5	,3199E-02	-,3777E-02	,3286E-02	-,3646E-02
	10	,4635E-02	-,4961E-02	,4814E-02	-,4656E-02
	20	,6266E-02	-,7042E-02	,6833E-02	-,6280E-02
	30	,7269E-02	-,8780E-02	,8340E-02	-,7455E-02

scale of irregularities, the formulas (52), (53), (54) for an effective impedance result in necessity to summarize series having Clebsch-Gordan's coefficients, which is actually a specific character of just a spherical problem.

Appendix

Calculation of $O_{\pm 5n}$ and $O_{\pm 6n}$

The transition from natural spherical basis $\hat{i}_r, \hat{i}_\theta, \hat{i}_\varphi$ to spiral one is provided with relations

$$\hat{i}_+ = -\frac{1}{\sqrt{2}}\left(\hat{i}_\varphi + i\,\hat{i}_\theta\right), \qquad \hat{i}_- = \frac{1}{\sqrt{2}}\left(\hat{i}_\varphi - i\,\hat{i}_\theta\right), \quad \hat{i}_0 = \hat{i}_r. \tag{A1}$$

Replacement of Legendre's joined functions $P_n^m(\cos\theta)$ with generalized spherical ones $P_{m,m'}^n(\cos\theta)$

$$\begin{cases} \dfrac{d}{d\theta}P_n^m(\cos\theta) = \dfrac{1}{2}i^{m+1}\sqrt{n(n+1)}\times \\ \qquad \times\left(P_{m,1}^n(\cos\theta) + P_{m,-1}^n(\cos\theta)\right), \\ \dfrac{m}{\sin\theta}P_n^m(\cos\theta) = \dfrac{1}{2}i^{m+1}\sqrt{n(n+1)}\times \\ \quad \times\sqrt{\dfrac{(n+m)!}{(n-m)!}}\left(P_{m,1}^n(\cos\theta) - P_{m,-1}^n(\cos\theta)\right), \end{cases} \tag{A2}$$

:rives the wave spherical harmonics $\vec{m}_{nm}^{(e,o)}$ and $\vec{n}_{nm}^{(e,o)}$ to a form

$$\begin{cases} \vec{m}_{nm}^{(e,o)} = \dfrac{Q_{nm}^{(e,o)}}{\sqrt{2}}Z_n(kr)\sqrt{\dfrac{4\pi}{2n+1}}\left\{\hat{i}_{nm}^+ - \hat{i}_{nm}^-\right\}, \\ \vec{n}_{nm}^{(e,o)} = \dfrac{-iQ_{nm}^{(e,o)}}{kr}\sqrt{\dfrac{4\pi n(n+1)}{2n+1}}\cdot\hat{i}_{nm}^r + \\ \qquad + \dfrac{Q_{nm}^{(e,o)}}{\sqrt{2}}\sqrt{\dfrac{4\pi}{2n+1}}\cdot\left\{\hat{i}_{nm}^+ + \hat{i}_{nm}^-\right\}, \end{cases} \tag{A3}$$

where $Q_{nm}^{(e,o)}$ are defined after the formula (7).

Scalar product of vectorial functions $\hat{i}_{nm}^+$ and $\hat{i}_{n'm'}^+$ is defined as follows [4]:

$$\left(\hat{i}_{nm}^\pm * \bar{\hat{i}}_{n'm'}^\pm\right) = \int_0^\pi d\theta\,\sin\theta\int_0^{2\pi}d\varphi\; t_{m,-1}^n(-\varphi,\theta,0)\,\bar{t}_{m',-1}^{n'}(-\varphi,\theta,0)\times\left(\hat{i}^\pm\cdot\bar{\hat{i}}^\pm\right)\cdot\sqrt{\frac{(2n+1)(2n'+1)}{4\pi}}. \tag{A4}$$

The horizontal line above $\bar{\hat{i}}_{nm}^\pm$ means complex conjugation, which converts contravariant components of the spiral basis into covariant ones [5], thereby $\left(\hat{i}^+\cdot\bar{\hat{i}}^+\right) = \left(\hat{i}^-\cdot\bar{\hat{i}}^-\right) = 1$ and $\left(\hat{i}^+\,\bar{\hat{i}}^-\right) = 0$. By virtue of it and of orthogonality of the matrix elements $t_{m,m'}^n(-\varphi,\theta,0)$ [4,5]:

$$\left(\hat{i}_{nm}^+ * \bar{\hat{i}}_{n'm'}^+\right) = \left(\hat{i}_{nm}^- * \bar{\hat{i}}_{n'm'}^-\right) = \delta_{m,m'}\,\delta_{n,n'}, \qquad \left(\hat{i}_{nm}^+ * \bar{\hat{i}}_{n'm'}^-\right) = 0. \tag{A5}$$

According to expansion into the Clebsch-Gordan's series ([4], Chapter3)

$$t_{j,j'}^{l_1}(-\varphi,\theta,0)\cdot t_{k,k'}^{l_2}(-\varphi,\theta,0) = \sum_{l=|l_1-l_2|}^{l_1+l_2}C(l_1,l_2,l;j,k,j+k)\times C(l_1,l_2,l;j',k',j'+k')\cdot t_{j+j',k+k'}^l(-\varphi,\theta,0). \tag{A6}$$

Proceeding from expression (7) for $\zeta(\varphi,\theta)$, we have

$$\vec{\gamma} = \nabla_\perp\zeta(\varphi,\theta) = -\frac{1}{2\sqrt{2}r}\times\sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\zeta_{n_4m_4}^{(e,o)}\frac{\sqrt{n_4(n_4+1)}}{N_{n_4m_4}}\cdot\left[\hat{i}_{m_4m_4}^+ + \hat{i}_{n_4m_4}^-\right]. \tag{A7}$$

From this,

$$\left\langle\gamma^2\right\rangle\Big|_{r=a} = -\frac{1}{8a^2}4\pi\left(\frac{\sigma}{a}\right)^2\times\sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}(-1)^{m_4}(2n_4+1)\times\widetilde{W}(n_4/a)n_4(n_4+1)\times\sum_{l=|n_4-n_1|}^{n_4+n_1}C(n_4,n_1,l;m_4,-m_4,0)\times$$

$$\times\left[C(n_4,n_1,l;-1,1,0)+C(n_4,n_1,l;1,-1,0)\right]\times$$
$$\times t^{l}_{0,0}(-\varphi,\theta,0)\cdot\delta_{n_4,n_1}\delta_{-m_4,m_1}. \qquad (A8)$$

Obukhov's theorem for the correlator of spectral amplitudes [6,7] is used here

$$\left\langle\tilde{\zeta}^{(e,o)}_{n_1m_1}\cdot\tilde{\zeta}^{(e,o)}_{n_4m_4}\right\rangle=(-1)^{m_4}(2n_4+1)\times$$
$$\times 4\pi\left(\frac{\sigma}{a}\right)^2\tilde{W}(n_4/a)\cdot\delta_{n_4,n_1}\delta_{-m_4,m_1}. \qquad (A9)$$

As

$$\mathcal{H}_\theta\mp i\mathcal{H}_\varphi=$$
$$=i\sqrt{\frac{\varepsilon}{\mu}}\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\alpha^{\mp(e,o)}_{n_3m_3}(x)\; t^{n_3}_{m_3,\pm1}(-\varphi,\theta,0), \qquad (A10)$$

where

$$\alpha^{\mp(e,o)}_{n_3m_3}=Q^{(e,o)}_{n_3m_3}\times$$
$$\times\left\{\frac{1}{x}\left(xZ_{n_3}(x)\right)'A^{(e,o)}_{n_3|m_3|}-(\mp)iB^{(e,o)}_{n_3|m_3|}Z_{n_3}(x)\right\}, \qquad (A11)$$

then

$$\vec{U}_6=\frac{\eta_0}{2\sqrt{2}}<\gamma^2>\left\{\hat{i}_-(\mathcal{H}_\theta-i\mathcal{H}_\varphi)-\hat{i}_+(\mathcal{H}_\theta+i\mathcal{H}_\varphi)\right\}=$$
$$=\frac{i\eta_0}{2\sqrt{2}}\frac{\pi}{2a^2}\left(\frac{\sigma}{a}\right)^2\sqrt{\frac{\varepsilon}{\mu}}\cdot\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\sum_{l=0}^{2n_4}\sum_{\tilde{l}=|l-n_3|}^{l+n_3}$$
$$(-1)^{m_4}(2n_4+1)\cdot\tilde{W}(n_4/a)n_4(n_4+1)\times$$
$$\times C(n_4,n_4,l;m_4,-m_4,0)\times$$
$$\times C(n_4,n_4,l;1,-1,0)\left[-(-1)^{l-n_4-n_4}+1\right]\times$$
$$\times C(n_3,l,\tilde{l};m_3,0,m_3)\times$$
$$\times C(n_3,l,\tilde{l};\mp1,0,\mp1)\cdot\alpha^{\pm(e,o)}_{n_3m_3}(\pm\hat{i}_\pm)\cdot t^{\tilde{l}}_{m_3,\mp1}(-\varphi,\theta,0). \qquad (A12)$$

Here, for abbreviation, it is designated

$$\alpha^{\pm}(\pm\hat{i}_\pm)(\cdots)=+\alpha^{+}\hat{i}_+(\cdots)-\alpha^{-}\hat{i}_-(\cdots).$$

Correspondingly, for

$$O_{\pm6n}=$$
$$=-\frac{N_{nm}}{\sqrt{2}}\left[\left(\vec{U}_6*\bar{\hat{i}}^{+}_{nm}\right)\pm\left(\vec{U}_6*\bar{\hat{i}}^{-}_{nm}\right)\right]\Big/\left(\begin{matrix}i\tilde{B}^{(e,o)}_{nm}\hat{Z}_n(x)\\ \tilde{A}^{(e,o)}_{nm}\hat{Z}'_n(x)\end{matrix}\right),$$

we obtain

$$O_{\pm6n}=\pm\frac{i\eta_0}{4a^2x}\left(\frac{\sigma}{a}\right)^2\times$$
$$\times\sqrt{\frac{\varepsilon}{\mu}}\,2\pi\sum_{n_4=0}^{\infty}(2n_4+1)n_4(n_4+1)\cdot\tilde{W}\left(\frac{n_4}{a}\right). \qquad (A13)$$

In deriving this result, the following relations ([5], Chapter8) were used:

$$\sum_{m_4=-n_4}^{n_4}(-1)^{m_4}C(n_4,n_4,l;m_4,-m_4,0)=(-1)^{n_4}\sqrt{2n_4+1}\,\delta_{l,0}$$
$$C(n,0,n;m,0,m)\equiv1,\quad C(n,0,n;\mp1,0,\mp1)\equiv1,$$
$$C(n_4,n_4,0;1,-1,0)=\frac{(-1)^{n_4}-1}{\sqrt{2n_4+1}}.$$

For the summand

$$\vec{U}_5=\frac{\eta_0}{\sqrt{2}}\left\langle\left\{\hat{i}_-(\gamma_\theta-i\gamma_\varphi)-\hat{i}_+(\gamma_\theta+i\gamma_\varphi)\right\}(\vec{\gamma}\cdot\vec{\mathcal{H}})\right\rangle$$

we, similarly, have

$$(\vec{\gamma}\cdot\vec{\mathcal{H}})\Big|_{r=a}=-\frac{1}{2\sqrt{2}a}\times$$
$$\times\sqrt{\frac{\varepsilon}{\mu}}\cdot\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}\tilde{\zeta}^{(e,o)}_{n_1m_1}\sqrt{n_1(n_1+1)}\times$$
$$\times\frac{1}{\sqrt{2}}Q^{(e,o)}_{nm}\left\{\left[A^{(e,o)}_{n_3|m_3|}\left(\frac{1}{x}\left(xZ_{n_3}(x)\right)'\right)+iB^{(e,o)}_{n_3|m_3|}Z_{n_3}(x)\right]\times\right.$$
$$\times t^{n_1}_{m_1,-1}(-\varphi,\theta,0)\,t^{n_3}_{m_3,+1}(-\varphi,\theta,0)+$$
$$+\left[A^{(e,o)}_{n_3|m_3|}\left(\frac{1}{x}\left(xZ_{n_3}(x)\right)'\right)-iB^{(e,o)}_{n_3|m_3|}Z_{n_3}(x)\right]\times$$
$$\left.\times t^{n_1}_{m_1,+1}(-\varphi,\theta,0)\,t^{n_3}_{m_3,-1}(-\varphi,\theta,0)\right\}, \qquad (A14)$$

$$\gamma_\theta\mp i\gamma_\varphi\Big|_{r=a}=\frac{i}{2a}\sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\tilde{\zeta}^{(e,o)}_{n_4m_4}\sqrt{n_4(n_4+1)}\times$$
$$\times\left[\hat{i}_-t^{n_4}_{m_4,+1}(-\varphi,\theta,0)-\hat{i}_+t^{n_4}_{m_4,-1}(-\varphi,\theta,0)\right].$$

As a result, we receive

$$\vec{U}_5=-\frac{i\eta_0}{8\sqrt{2}a^2}\times$$
$$\times\sqrt{\frac{\varepsilon}{\mu}}\sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}$$
$$\sum_{l=|n_1-n_3|}^{n_1+n_3}\sum_{\tilde{l}=|n_4-l|}^{n_4+l}\left\langle\tilde{\zeta}^{(e,o)}_{n_4m_4}\cdot\tilde{\zeta}^{(e,o)}_{n_1m_1}\right\rangle\sqrt{n_4(n_4+1)n_1(n_1+1)}\times$$

$$\times\left\{\left[\tilde{A}^{(e,o)}_{n_3m_3}\frac{1}{x}\left(xZ_{n_3}(x)\right)' + i\tilde{B}^{(e,o)}_{n_3m_3}Z_{n_3}(x)\right]\times\right.$$
$$\times C(n_1,n_3,l;-1,1,0)\times$$
$$\times\left[\hat{i}^-_{\tilde{l},m_4+m_1+m_3}\cdot C(n_4,l,\tilde{l};1,0,1)-\hat{i}^+_{\tilde{l},m_4+m_1+m_3}\times\right.$$
$$\left.\times C(n_4,l,\tilde{l};-1,0,-1)\right]+$$
$$+\left[\tilde{A}^{(e,o)}_{n_3m_3}\frac{1}{x}\left(xZ_{n_3}(x)\right)' - i\tilde{B}^{(e,o)}_{n_3m_3}Z_{n_3}(x)\right]\times$$
$$\times C(n_1,n_3,l;1,-1,0)\times$$
$$\times\left[\hat{i}^-_{\tilde{l},m_4+m_1+m_3}\cdot C(n_4,l,\tilde{l};1,0,1)-\hat{i}^+_{\tilde{l},m_4+m_1+m_3}\times\right.$$
$$\left.\left.\times C(n_4,l,\tilde{l};-1,0,-1)\right]\right\}\times$$
$$\times C(n_1,n_3,l;m_1,m_3,m_1+m_3)\times$$
$$\times C(n_4,l,\tilde{l};m_4,m_1+m_3,m_4+m_1+m_3)N^{-1}_{n_3m_3}. \qquad \text{(A15)}$$

Let us calculate scalar product

$$\left(\vec{U}_5*\hat{i}^+_{nm}\right)=\frac{i\eta_0}{8\sqrt{2}a^2}\times$$
$$\times\sqrt{\frac{\varepsilon}{\mu}}\cdot\sum_{n_4=0}^{\infty}\sum_{m_4=-n_4}^{n_4}\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}$$
$$\sum_{l=|n_1-n_3|}^{n_1+n_3}\sum_{\tilde{l}=|n_4-l|}^{n_4+l}4\pi\left(\frac{\sigma}{a}\right)^2(-1)^{m_4}(2n_4+1)\times$$
$$\times\tilde{W}(n_4/a)\,\delta_{n_4,n_1}\delta_{-m_4,m_1}\times\sqrt{n_4(n_4+1)n_1(n_1+1)}\times$$
$$\times C(n_1,n_3,l;m_1,m_3,m_1+m_3)\times$$
$$\times C(n_4,l,\tilde{l};m_4,m_1+m_3,m_4+m_1+m_3)\times$$
$$\times\left\{\left[\tilde{A}^{(e,o)}_{n_3m_3}\left(\frac{1}{x}\left(xZ_{n_3}(x)\right)'\right)+i\tilde{B}^{(e,o)}_{n_3m_3}Z_{n_3}(x)\right]\times\right.$$
$$\times C(n_1,n_3,l;-1,1,0)\times C(n_4,l,\tilde{l};-1,0,-1)+$$
$$+\left[\tilde{A}^{(e,o)}_{n_3m_3}\frac{1}{x}\left(xZ_{n_3}(x)\right)' - i\tilde{B}^{(e,o)}_{n_3m_3}Z_{n_3}(x)\right]\times$$
$$\left.\times C(n_1,n_3,l;1,-1,0)C(n_4,l,\tilde{l};-1,0,-1)\right\}\times$$
$$\times\delta_{n,\tilde{l}}\,\delta_{m,m_4+m_1+m_3}N^{-1}_{n_3m_3}. \qquad \text{(A16)}$$

Since $0\le n_4\le\infty, |m_4|\le n_4, 0\le n_1\le\infty, |m_1|\le n_1$, then at fixed n_4, there is always achievable $n_1=n_4$, so and $m_1=-m_4$, at which $\delta_{n_4,n_1}\delta_{-m_4,m_1}=1$.

Thus, summation with respect to n_1,m_1 results in retain of the summands at which $n_1=n_4$ and $m_1=-m_4$, thereby $\delta_{m,m_4+m_1+m_3}=\delta_{m,m_3}$. By virtue of Kponecker's symbols $\delta_{n,\tilde{l}}\cdot\delta_{m,m_3}$ being present in the cofactors, it is possible to replace $\tilde{l}$ with n, and m_3 with m at them. Interchanging the summation order in (16), we come to a sum of a form

$$SUM=(....,m_3=m,....)\times$$
$$\times\sum_{m_4}\sum_{m_3}(-1)^{m_4}C(n_4,n_3,l;-m_4,m_3,-m_4+m_3)\times$$
$$\times C(n_4,l,n;m_4,-m_4+m_3,m)\,\delta_{m,m_3}, \qquad \text{(A17)}$$

where $(....,m_3=m,....)$ means external summation with respect to all other variables, and in the summands, the substitution of m_3 with m is accomplished. Using property of symmetry of Clebsch-Gordan's coefficients, it is possible to receive

$$SUM=(....,m_3=m,....)(-1)^{n_3-l}\sqrt{\frac{2l+1}{2n_3+1}}\times$$
$$\times\sum_{m_4=-n_4}^{n_4}\sum_{m_3=-n_3}^{n_3}\delta_{m,m_3}C\,(n_4,l,n_3;m_4,-m_4+m_3,m)\times$$
$$\times C(n_4,l,n;m_4,-m_4+m_3,m_3)=$$
$$=(....,m_3=m,....)(-1)^{n_3-l}\times$$
$$\times\sqrt{\frac{2l+1}{2n_3+1}}\cdot\begin{cases}\delta_{n,n_3} & (|m|\le n_4+l),\\ 0 & (|m|>n_4+l).\end{cases} \qquad \text{(A18)}$$

Such result is a consequence of orthogonality of Clebsch-Gordan's coefficients [4,5]. Then

$$SUM=N^{-1}_{nm}\times$$
$$\times\sum_{n_4=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{l=|n_4-n_3|}^{n_4+n_3}\sum_{\tilde{l}=|n_4-l|}^{n_4+l}(....,m_3=m,n_3=n,....)\times$$
$$\times\delta_{n,n_3}\delta_{n,\tilde{l}}(-1)^{n-l}\sqrt{\frac{2l+1}{2n+1}}\cdot\begin{cases}1 & (|m|\le n_4+l),\\ 0 & (|m|>n_4+l).\end{cases} \qquad \text{(A19)}$$

In view of presence of δ_{n,n_3}, in the cofactors, replacement of $n_3\to n$ is accomplished everywhere, as the intervals of change of n and n_3 are identical $[0,\infty]$.

The condition of $|m| \le n_4 + l$ excludes of the summation a triangular domain near to the top of the first quadrant with the sides of $n_4 = m$ and $l = m$ along the axes n_4 and l. The summation with respect to $\tilde{l}$ cuts of this domain a half-band being rested on the axes $\tilde{l}$ and n_4 in the points $\tilde{l} = n$ and $n_4 = n$, and extending along the first quadrant bisectrix. For the top half-band part that lies above the bisectrix, the limitations

$$n_4 \le l \le n_4 + n, \quad n - l \le n_4 \tag{A20}$$

are satisfied. For the lower band part that lies under the bisectrix, the limitations on l and n_4 take the form

$$n_4 - n \le l \le n_4, \quad n - l \le n_4. \tag{A21}$$

Proceeding from (A 20) and (A 21), it is rather easy to establish limitations on the upper bound $\tilde{l}_{\max} = n_4 + l$ of summation with respect to $\tilde{l}$ and on the lower one $\tilde{l}_{\min}$, which is equal to $\tilde{l}_{\min} = l - n_4$ in the top half-band part and $\tilde{l}_{\min} = n_4 - l$ in the lower half-band part of the summation with respect to (l, n_4)

$$n \le \tilde{l}_{\max}, \qquad \tilde{l}_{\min} \le n. \tag{A22}$$

Thus, the summation with respect to $\tilde{l}$ always contains the point $\tilde{l} = n$ for which $\delta_{\tilde{l},n} = 1$. As the result, the initial 8-fold sum is reduced to 2-fold one.

$$SUM = N_{nm}^{-1} \sum_{n_4=0}^{\infty} \sum_{l=|n_4-n|}^{n_4+n} (-1)^{n-l} \times$$
$$\times \sqrt{\frac{2l+1}{2n+1}} (\ldots, m_3 = m, n_3 = n, \ldots). \tag{A 23}$$

As a result of similar calculations,

$$\left(\vec{U}_5 * \bar{\hat{i}}^{\pm}_{nm}\right) = \pm \frac{\pi i \eta_0}{2\sqrt{2} N_{nm} a^2} \left(\frac{\sigma}{a}\right)^2 \times$$
$$\times \sqrt{\frac{\varepsilon}{\mu}} \cdot \sum_{n_4=0}^{\infty} \sum_{l=|n-n_4|}^{n+n_4} (2n_4+1) \times$$
$$\times n_4(n_4+1) \widetilde{W}(n_4/a) (-1)^{n-l} \times$$
$$\times \sqrt{\frac{2l+1}{2n+1}} \left\{ \left[\widetilde{A}^{(e,o)}_{nm} \frac{1}{x} (x Z_n(x))' + i\widetilde{B}^{(e,o)}_{nm} Z_n(x) \right] \times \right.$$
$$\times C(n_4, n, l; -1, 1, 0) + \left[\widetilde{A}^{(e,o)}_{nm} \frac{1}{x} (x Z_n(x))' - i\widetilde{B}^{(e,o)}_{nm} Z_n(x) \right] \times$$
$$\left. \times C(n_4, n, l; 1, -1, 0) \right\} C(n_4, n, l; \mp 1, 0, \mp 1). \tag{A 24}$$

Using recurrent relations and symmetry properties of Clebsch-Gordan's coefficients, the expression (A24) can be transformed to a form containing only $C(n, n_4, l; 0,0,0)$ and $C(n, n_4, l; 1,0,1)$. Thereby,

$$O_{\pm 5n} = \frac{N_{nm}}{\sqrt{2}} U_{\pm 5nm} \Big/ \begin{pmatrix} i\widetilde{B}^{(e,o)}_{nm} \hat{Z}_n(x) \\ \widetilde{A}^{(e,o)}_{nm} \hat{Z}'_n(x) \end{pmatrix} =$$
$$= \frac{N_{nm}}{\sqrt{2}} \left\{ \left(\vec{U}_5 \cdot \bar{\hat{i}}^{+}_{nm}\right) \pm \left(\vec{U}_5 \cdot \bar{\hat{i}}^{-}_{nm}\right) \right\} \Big/ \begin{pmatrix} i\widetilde{B}^{(e,o)}_{nm} \hat{Z}_n(x) \\ \widetilde{A}^{(e,o)}_{nm} \hat{Z}'_n(x) \end{pmatrix} =$$
$$= -\frac{\pi i \eta_0}{2a^2 x} \left(\frac{\sigma}{a}\right)^2 \sqrt{\frac{\varepsilon}{\mu}} \times$$
$$\times \sum_{n_4=0}^{\infty} \sum_{l=|n-n_4|}^{n+n_4} (2n_4+1) n_4 (n_4+1) \widetilde{W}(n_4/a) \times$$
$$\times \left\{ (-1)^{\alpha} \sqrt{\frac{l(l+1)}{n_4(n_4+1)}} C(n, n_4, l; 1,0,1) - \right.$$
$$\left. - \sqrt{\frac{n(n+1)}{n_4(n_4+1)}} C(n, n_4, l; 0,0,0) \right\} \times$$
$$\times \left\{ \sqrt{\frac{l(l+1)}{n_4(n_4+1)}} C(n, n_4, l; 1,0,1) - \right.$$
$$\left. - \sqrt{\frac{n(n+1)}{n_4(n_4+1)}} C(n, n_4, l; 0,0,0) \right\} \times \left[1 \mp (-1)^{\alpha}\right], \tag{A25}$$

where $\alpha = n - n_4 - l$. The top line $i\widetilde{B}^{(e,o)}_{nm} Z_n(x)$ in the parentheses corresponds to U_{+5nm}, and the lower one $\widetilde{A}^{(e,o)}_{nm} \frac{1}{x} (x Z_n(x))'$ corresponds to U_{-5nm}. Note that the factor $1 \mp (-1)^{\alpha} = 1 \mp (-1)^{n-n_4-l}$ is different from zero for odd $\alpha = n - n_4 - l$ for the upper sign, and for even $\alpha = n - n_4 - l$ for the lower one.

Transition from summation to integration at $n, n_4, l >> 1$

The transition to integration at $n_4 >> 1$ in expression (A 13) for $O_{\pm 6n}$ is carried out rather easily: it is necessary to convert from integer n_4 with discrete change of $\Delta n_4 = 1$ to wave number $\chi = n_4/a$ and $d\chi = 1/a$, thereby $2n_4 + 1 \approx 2\chi a$, $n_4(n_4+1) \approx n_4^2 = \chi^2 a^2$.
Then

$$O_{\pm 6n} = \pm i\eta_0 \times$$
$$\times \sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{1}{4a^2}\left(\frac{\sigma}{a}\right)^2 \frac{2\pi}{x} \int_0^\infty 2\chi^3 a^4 \, \tilde{W}(\chi) d\chi =$$
$$= \pm i\eta_0 \sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{\sigma^2}{2x} \int_0^\infty \chi \, d\chi \, \tilde{W}(\chi) \chi^2 \int_0^{2\pi} d\varphi . \tag{A 26}$$

In conversion to integration in expression (A 25) for $O_{\pm 5n}$, it is necessary to take into account the presence, under the sum sign, of the "flickering" factor $1 \pm (-1)^{n-n_4-l}$, because of which the non-zero values in the internal sum with respect to l occur within $\Delta l = 2$. Therefore,

$$SUM = \sum_{n_4=0}^{\infty} \sum_{l=|n-n_4|}^{n+n_4} \mathrm{A}(n, n_4, l) =$$
$$= \frac{1}{2} a^2 \sum_{\chi=0}^{\infty} d\chi \sum_{q=\frac{|n-n_4|}{a}}^{\frac{n+n_4}{a}} dq \, \mathrm{A}(n, \chi a, qa). \tag{A 27}$$

According to the vectorial model of addition of the moments [5], l is a module of the vector being the vectorial sum of $\vec{n}$ and $\vec{n}_4$ forming an angle φ, i.e.

$$l = \sqrt{n^2 + n_4^2 + 2nn_4 \cos\varphi} ,$$
$$q = a^{-1} \sqrt{n^2 + n_4^2 + 2nn_4 \cos\varphi} ,$$

whence

$$-d\varphi = \frac{2a^2 q dq}{\sqrt{4n_4^2 n^2 - \left(q^2 a^2 - n^2 - n_4^2\right)^2}} =$$
$$= \frac{2al dq}{\sqrt{4n_4^2 n^2 - \left(l^2 - n^2 - n_4^2\right)^2}} .$$

Taking into account that $|n - n_4|/a$ corresponds to $\varphi = \pi$, and $(n + n_4)/a$ corresponds to the value of $\varphi = 0$, we obtain $(n_4 = \chi a)$

$$SUM = \tfrac{1}{4} a \int_0^\infty d\chi \int_0^\pi d\varphi \; l^{-1} \mathcal{A}(n, n_4, l) \times$$
$$\times \sqrt{4n^2 n_4^2 - \left(l^2 - n^2 - n_4^2\right)^2} \Bigg|_{\substack{l=\sqrt{n^2+\chi^2 a^2 + 2n\chi a \cos\varphi} \\ n_4 = \chi a}} . \tag{A28}$$

As a result, for $O_{\pm 5n}$ we receive:

$$O_{\pm 5n} = -i\eta_0 \sqrt{\frac{\varepsilon}{\mu}} \frac{\sigma^2}{x} \int_0^\infty d\chi \tilde{W}(\chi) \chi^3 \int_0^{2\pi} d\varphi \begin{pmatrix} -\sin^2\varphi \\ +\cos^2\varphi \end{pmatrix} .$$

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