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Electromagnetic Step Signal Propagation in Lossy Waveguide

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Propagation of the transient electromagnetic field in a waveguide with the perfectly conducting surface of arbitrary cross-section which is filled with a lossy medium is considered. Time Domain Method of Waveguide Evolutionary Equations has been used, and original initial-boundary value vector problem for Maxwell's equations has been reduced to integration of the one-dimensional Klein-Gordon scalar equation. Its solution is obtained as an expansion in terms of a suitable for Time Domain Method basic set. The expansion coefficients are readily derived through the same decomposition of the input signal. Analytical forms of early- and late- time approximations for propagating electromagnetic signal are presented, and the validity conditions for the exact solution are discussed.

1. Introduction

Present practice in transmission of the information depends on propagation of transient superwideband electromagnetic waves, or signals, for short. Solution to the problem for signal in the Time Domain has priority over that for infinitely extended sinusoidal waves in, for instance, radar or stealth technology. Waveguides are used as important constituents of apparatus employed in these fields. It is obviously, that new approach should be developed for electromagnetic signal excitation and propagation analysis which allows for causality low.

As for background history of the problem, a reader ought to be referred to paper [1] where one can find rather complete list of previous publications. The list of references should be complemented by the several publications [2-5] which have been omitted there. This paper presents a method for analytical investigation of the electromagnetic signal excitation and propagation through a waveguide.

A waveguide signal description has been traditionally based on the conventional concept of harmonic waves and usually expressed by applying Fourier Transform Techniques. But signal field approximation in terms of a superposition of harmonic waves does not enable to give us a clear idea how an electromagnetic wave with an initial time variation, (e.g., a step pulse), changes while propagating in a lossy waveguide. This is due to the fact that harmonic waves exist and are nonzero in the infinite intervals $-\infty < t,\ z < +\infty$. Therefore, they cannot provide relevant information regarding the time evolution of the signal which always has the start and the end and is irregular or discontinuous in time.

As it is seen, for instance, from [1,2], more powerful methods are the Time Domain Methods. This paper takes a new look at the propagation of signals in waveguides by applying Time Domain techniques. More precisely, Waveguide Evolutionary Equations [3] and Separation of Variables regarding to Group theory [6] are used.

Waveguide Evolutionary Equation method is a general method for the description of EM fields in waveguides. It assumes geometrically regular waveguide in the z direction of an arbitrary smooth cross section S, a contour L is the boundary of the cross section surface, $\vec{\nu}$ is the outer normal to the L, $\vec{z}_0$ is a unit vector along the z-axis, $\vec{r}$ is a two-dimensional position vector within the region of waveguide cross section, i.e. $\vec{r} \in S$.

The bounding surface of a waveguide is assumed to be perfectly conducting and single connected but there are no restrictions on the medium filling the waveguide, i.e., generally, the waveguide may be filled in with an linear inhomogenous, time-varying and even nonlinear medium. The complete definition of waveguide EM field given by [3] may be summed up as follows:

H- waves

$$\vec{H}_m(\vec{r},z,t) = \nabla_t \Psi_m(\vec{r}) \left(\frac{1}{\mu} \frac{\partial}{\partial z} \mu h_m(z,t) - \frac{1}{\mu_0 \mu} g_m \right);$$

$$H_{zm}(\vec{r},z,t) = \Psi_m(\vec{r}) p_m h_m(z,t);$$

$$\vec{E}_m(\vec{r},z,t) = \left[\vec{z}_0 \times \nabla_t \Psi_m \right] \times$$

$$\times \left(\mu_0 \frac{\partial}{\partial t} \mu\, h_m(z,t) + i_{zm} \right); \qquad (1)$$

$$E_{zm}(\vec{r},z,t) \equiv 0;$$

$$\left(\nabla_t^2 + p_m \right) \Psi_m(\vec{r}) = 0; \quad \frac{\partial}{\partial \nu} \Psi_m \Big|_L = 0;$$

$$\vec{r} \in S; \quad \frac{p_m}{S} \int_S dS \left| \Psi_m \right|^2 = 1; \qquad (2)$$

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$$\left(\varepsilon_0\mu_0\frac{\partial}{\partial t}\varepsilon\frac{\partial}{\partial t}\mu-\frac{\partial}{\partial z}\frac{1}{\mu}\frac{\partial}{\partial z}\mu+p_m\right)h_m(z,t)=$$

$$=j_m-\varepsilon_0\frac{\partial}{\partial t}\left(\varepsilon\, i_{zm}\right)-\frac{1}{\mu_0}\frac{\partial}{\partial z}\frac{1}{\mu}g_m\;; \qquad (3)$$

E- waves

$$\vec{E}_n(\vec{r},z,t)=$$

$$=\nabla_t\Phi_n(\vec{r})\left(\frac{1}{\varepsilon}\frac{\partial}{\partial z}\varepsilon\, e_n(z,t)-\frac{1}{\varepsilon_0\varepsilon}\rho_n\right);$$

$$E_{zn}(\vec{r},z,t)=\Phi_n(\vec{r})q_n\, e_n(z,t); \qquad (4)$$

$$\vec{H}_n(\vec{r},z,t)=-\left[\vec{z}_0\times\nabla_t\Phi_n\right]\times$$

$$\times\left(\varepsilon_0\frac{\partial}{\partial t}\varepsilon\, e_n(z,t)-j_{zn}\right);$$

$$H_{zn}(\vec{r},z,t)\equiv 0\,;$$

$$\left(\Delta_t+q_n\right)\Phi_n(\vec{r})=0;\quad \Phi|_L=0;$$

$$\vec{r}\in S;\quad \frac{q_n}{S}\int_S dS\left|\Phi_n\right|^2=1; \qquad (5)$$

$$\left(\varepsilon_0\mu_0\frac{\partial}{\partial t}\mu\frac{\partial}{\partial t}\varepsilon-\frac{\partial}{\partial z}\frac{1}{\varepsilon}\frac{\partial}{\partial z}\varepsilon+q_n\right)e_n(z,t)=$$

$$=i_n-\mu_0\frac{\partial}{\partial t}\left(\mu j_{zn}\right)-\frac{1}{\varepsilon_0}\frac{\partial}{\partial z}\frac{1}{\varepsilon}\rho_n. \qquad (6)$$

Here $\nabla_t=\nabla-\vec{z}_0\frac{\partial}{\partial z}$ is transverse delta-operator.

Eqs. (1) and (4) represent decomposition of the five-component EM fields in a waveguide into transverse $\vec{E},\vec{H}$ and longitudinal E_z, H_z components. In the case that the waveguide sustains a lot of modes the total field is given by a sum over all possible integers n or m.

Membrane functions $\Psi_m(\vec{r})$ and $\Phi_n(\vec{r})$ determine a transverse spatial distribution of the waveguide EM fields. They are defined in terms of the boundary- eigenvalue Neumann and Dirichlet problems in Eqs. (2) and (5), respectively. Here $\nabla_t^2=\Delta-\vec{z}_0\frac{\partial^2}{\partial z^2}$ is the transverse Laplacian; p_m and q_n are positive real eigenvalues defined by the indexes n, $m=0,1,2\ldots$.

The scalar functions $h_m(z,t)$ and $e_n(z,t)$ completely determine field evolution along the z - axis. Mathematically, they represent evolutionary coefficients of the EM field expansion in transverse vector functions and, physically, an amplitude of longitudinal magnetic or electric field components. Coefficients $h_m(z,t)$ and $e_n(z,t)$ to be found satisfy Evolutionary Waveguide Equations (3), (6) with predetermined initial and (or) boundary conditions.

Everywhere constants ε_0 and μ_0 are the conversion factors in MKS rationalized units; $\varepsilon\equiv\varepsilon(z,t)$ and $\mu\equiv\mu(z,t)$ functions are permittivity and permeability of medium, respectively. The right hand sides of Eqs. (1), (3), (4), (6) contain six scalar functions $j_m, j_{zn}, g_m, \rho_n, i_{zm}, i_n$. They all are defined by the constitutive relations and the impressed force functions in accordance with the following definitions:

$$j_m(z,t)=\frac{1}{S}\int_S dS\left(\vec{j}_0+\vec{j}_\sigma+\frac{\partial}{\partial t}\vec{\boldsymbol{P}}'\right)\left[\nabla_t\Psi_m^*\times\vec{z}_0\right];$$

$$i_n(z,t)=\frac{1}{S}\int_S dS\left(\mu_0\frac{\partial}{\partial t}\boldsymbol{M}_z'\right)\Psi_m^*;$$

$$j_{zn}(z,t)=\frac{1}{S}\int_S dS\left(j_{0z}+j_{\sigma z}+\frac{\partial}{\partial t}\boldsymbol{P}_z'\right)\Phi_n^*;$$

$$i_{zm}(z,t)=\frac{1}{S}\int_S dS\left(\mu_0\frac{\partial}{\partial t}\left(\vec{z}_0\boldsymbol{M}_z'\right)\right)\left[\vec{z}_0\times\nabla_t\Phi_n^*\right]; \qquad (7)$$

$$g_m(z,t)=\frac{1}{S}\int_S dS\left(-\mu_0\, div\left(\vec{z}_0\,\boldsymbol{M}_z'\right)\right)\Psi_m^*;$$

$$\rho_n(z,t)=\frac{1}{S}\int_S dS\left(\rho_0+\rho_\sigma-div\left(\vec{z}_0\,\boldsymbol{P}_z'\right)\right)\Phi_n^*,$$

where ρ_σ and j_σ are the density and current density of free charges, ρ_0 and j_0 are corresponding functions of the impressed forces; $\vec{\boldsymbol{P}}', \boldsymbol{P}_z'$ and $\vec{\boldsymbol{M}}', \boldsymbol{M}_z'$ are transverse and longitudinal nonlinear parts of polarization $\vec{\mathcal{P}}$ and magnetization $\vec{\mathcal{M}}$ vectors, respectively. These vectors have been defined in [3] as follows:

$$\vec{\mathcal{P}}=\varepsilon_0\left(\varepsilon(z,t)-1\right)\vec{\mathcal{E}}+\vec{\mathcal{P}}'(\vec{\mathcal{E}},\vec{\mathcal{H}});$$

$$\vec{\mathcal{M}}=\left(\mu(z,t)-1\right)\vec{\mathcal{H}}+\vec{\mathcal{M}}'(\vec{\mathcal{H}},\vec{\mathcal{E}});$$

$$\vec{\mathcal{E}}=\vec{E}+\vec{z}_0E_z;$$

$$\vec{\mathcal{H}}=\vec{H}+\vec{z}_0H_z.$$

The solution to the problem for EM signals in lossy waveguide is obtained as an example of integration of evolutionary equations (3) and (6). In this case EM field is assumed to be absent in the whole volume of waveguide before fixed time $t=0$. Since the instant $t>0$ nonzero field originates in the cross- section at $z=0$ (step in the EM field) and changes with time in a predetermined fashion. We investigate the signal propagation problem analytically and numerically.

The present paper is organized as follows. In the second section of the paper the formulation of the problem for the arbitrary EM signal in the waveguide is shown. The third section is devoted to the discussion of the energy transport for the electromagnetic signal in the waveguide. In the forth section the exact solution of the problem for arbitrary signal propagation is obtained. The fifth section demonstrate the solution of the problem for case of electromagnetic step signal. At last in the sixth section of the paper we discuss the numerical and graphical results obtained and present the approximation of the exact solution.

2. Formulation of Problem for Waveguide EM Signal

We shall demonstrate the solution to the EM signal problem for a particular case of a waveguide with ohmic losses in absence of impressed forces when

$$\varepsilon\ (z,t) = const, \quad \mu\ (z,t) = const$$
$$\vec{\mathcal{P}}\,' = \vec{\mathcal{M}}\,' = 0;\ \vec{J}_0 = 0, \quad \rho_0 = 0;$$
$$\vec{J}_\sigma = \vec{j}_\sigma + \vec{z}_0 j_{\sigma z} = \sigma_1\left(\vec{E} + \vec{z}_0 E_z\right), \tag{8}$$
$$\sigma_1 = const.$$

With the conditions of (7), Eqs. (3) and (6) acquire the following form

$$\left(\varepsilon_0\mu_0\varepsilon\mu\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + p_m\right) h_m(z,t) = j_m(z,t)\ ; \tag{9}$$

$$\left(\varepsilon_0\mu_0\varepsilon\mu\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + q_n\right) e_n(z,t) =$$
$$= -\mu_0\mu\frac{\partial}{\partial t} j_{zn}(z,t) - \frac{1}{\varepsilon_0\varepsilon}\frac{\partial}{\partial z}\rho_n(z,t). \tag{10}$$

Let us calculate the right-hand sides of Eqs. (9), (10). Usage of definitions (7), and substitution expressions for $\vec{E}_m$ and E_{zn} from (1), (4) and (8) with regard to normalization in (2) and (5) yields

$$j_m(z,t) = \frac{1}{S}\int_S dS\left(\sigma\vec{E}_m\cdot\left[\nabla_t\,\Psi_m^*\times\vec{z}_0\right]\right) =$$
$$= -\sigma_1\,\mu_0\mu\frac{\partial}{\partial t}h_m(z,t); \tag{11}$$

$$j_{zn}(z,t) = \frac{1}{S}\int_S dS\,\sigma E_{zn}\Phi_n^* = \sigma_1 e_n(z,t)\ . \tag{12}$$

Since at every instant $t \le 0$ fields are assumed to be absent throughout the waveguide, i.e., $\vec{\mathcal{E}}\big|_{t\le 0} = 0$, under the condition of the problem, then $div\vec{\mathcal{E}}\big|_{t\le 0} = 0$. Such initial condition together with the continuity equations $div\vec{J}_\sigma = -\frac{\partial\rho_\sigma}{\partial t}$ and Coulomb law $\varepsilon\, div\vec{\mathcal{E}} = \rho_\sigma$ yield $\rho_\sigma(z)\big|_{0\le t<\infty} \equiv 0$. Thus, taking into account the initial condition, in Eq. (10) we have

$$\rho_n(z,t) = 0\,. \tag{13}$$

We now can state an initial and boundary-value waveguide problem for an EM waveguide signal. Inserting conditions (11), (12) into Eqs. (9), (10) and supplementing it by two point boundary specifications along with the null initial values yields

$$\begin{cases} \left(\varepsilon_0\mu_0\varepsilon\mu\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial z^2} + \mu_0\mu\sigma_1\dfrac{\partial}{\partial t} + \kappa^2\right) F(z,t) = 0; & (14) \\ F(z,t)\big|_{z=0} = H(t)\varphi(t), \quad \left|F(z,t)\right|\big|_{z\to\infty} < \infty, \quad 0 < t < \infty; & (15) \\ F(z,t)\big|_{t\le 0} = \dfrac{\partial}{\partial t}F(z,t)\Big|_{t\le 0} = \dfrac{\partial}{\partial z}F(z,t)\Big|_{t\le 0} = 0, \quad 0 \le |z| < \infty; & (16) \end{cases}$$

where $F(z,t) = h_m(z,t)$ or $e_n(z,t)$, and $\kappa^2 = p_m$ or q_n respectively; a standard nomenclature for the asymmetric step function have been used as follows

$$H(t) = \begin{cases} 1, & z = 0,\ t \ge 0; \\ 0, & -\infty < z < \infty, t < 0. \end{cases}$$

Problem (14), (15), (16) admits the damped solution

$$F(z,t) = f(z,t)e^{-\alpha t} \tag{17}$$

provided that $\alpha = \frac{\sigma_1}{2\varepsilon_0\varepsilon}$ and $f(z,t)$ satisfy corresponding initial and boundary problem for one-dimensional Klein-Gordon equation

$$\begin{cases} \left(\dfrac{\partial^2}{\partial\tau^2}-\dfrac{\partial^2}{\partial\zeta^2}+\tilde{\kappa}^2\right)f(\zeta,\tau)=0, \quad 0\le|\zeta|\le\tau<\infty; & (18)\\ f(0,\tau)=H(\tau)e^{\sigma\tau}\varphi(\tau), \quad |f(\zeta,\tau)|\big|_{\zeta\to\infty}<\infty, \quad 0<|\tau|<\infty; & (19)\\ f(\zeta,\tau)\big|_{\tau\le0}=\dfrac{\partial}{\partial\tau}f(\zeta,\tau)\big|_{\tau\le0}=\dfrac{\partial}{\partial\zeta}f(\zeta,\tau)\big|_{\tau\le0}=0, \quad 0\le|\zeta|<\infty; & (20) \end{cases}$$

where new dimensionless variables have been already introduced, namely,

$\tau=\dfrac{\kappa t}{\sqrt{\varepsilon_0\mu_0\varepsilon\mu}}$ (dimensionless time);

$\sigma=\dfrac{1}{2}\sqrt{\dfrac{\mu_0}{\varepsilon_0}\dfrac{\mu}{\varepsilon}}\dfrac{\sigma_1}{\kappa}$ (dimensionless conductivity); (21)

$\zeta=\kappa z$ (dimensionless distance);

$\tilde{\kappa}^2=1-\sigma^2$.

Should recall that $\kappa>0$ is the transverse waveguide number related to a waveguide cutt-off wavelength λ_c by $\kappa=2\pi\sqrt{\varepsilon\mu}/\lambda_c$.

3. Energy Transport by Waveguide EM Signals

Let us define energy $W(z,t)$ and longitudinal component of energy flux $P_z(z,t)$ (integrated over waveguide cross-section S and normalized on it) of waveguide EM signal

$$W(z,t)=\frac{1}{S}\times$$
$$\times\int_S\frac{1}{2}\left\{\varepsilon\varepsilon_0E^2(z,t)+\mu\mu_0H^2(z,t)\right\}dS, \qquad (22)$$

$$P_z(z,t)=\frac{1}{S}\int_S\left[E(z,t)\times H(z,t)\right]_z dS. \qquad (23)$$

Since the ratio of energy flux P_z and energy density W is a quantity having the dimensionality of a velocity we can define

$$v_S(z,t)=\frac{P_z(z,t)}{W(z,t)} \qquad (24)$$

as a velocity of energy propagation through waveguide. By contrast to the time (period) averages which are used in the description of the sinusoidal waves we need in Eqs. (22), (23), (24) to operate on momentary values in spite of rapid aperiodic time variation in all signal features. On insertion of the Eqs. (1), (4) into Eqs. (22), (23), (24) after the integration over cross-section with regard to normalization in (2), (5) we obtain in dimensionless variables

$$\frac{v_S(\zeta,\tau)}{v_l}=\begin{cases} 2\left(2\sigma F-\dfrac{\partial F}{\partial\tau}\right)\dfrac{\partial F}{\partial\zeta}\Big/\left[\left(2\sigma F-\dfrac{\partial F}{\partial\tau}\right)^2+\left(\dfrac{\partial F}{\partial\zeta}\right)^2+F^2\right], & (E-\text{waves})\\ -2\dfrac{\partial F}{\partial\tau}\dfrac{\partial F}{\partial\zeta}\Big/\left[\left(\dfrac{\partial F}{\partial\tau}\right)^2+\left(\dfrac{\partial F}{\partial\zeta}\right)^2+F^2\right], & (H-\text{waves}) \end{cases} \qquad (25)$$

where $v_l=1/\sqrt{\varepsilon\varepsilon_0\mu\mu_0}$ is the velocity of light in given medium. If σ is nonzero we have, by contrast to lossless medium, differences in energy features for the EM waveguide signal to be transported by E-waves and H-waves.

4. Solution to Problem for Waveguide EM Signal

One can readily see that there are three possible types of Eq. (18) depending on whether the conductivity σ is greater or less or equal to unity. In the latter case we have $\tilde{\kappa}=0$ which yields a one-dimensional wave equation. Its solution in a conventional form of traveling waves with Eq. (17) gives

$$F(\zeta,\tau)=e^{-|\zeta|}H(\tau-|\zeta|)\varphi(\tau-|\zeta|). \qquad (26)$$

Thus, the signal $H(\tau)\varphi(\tau)$ in such lossy waveguide moves away from the origin in its true shape and decays exponentially in its size. If $0\le\sigma<1$ or $1<\sigma<\infty$ we have in (18) conventional Klein-Gordon equation for $f(\zeta,\tau)$ or $f(i\zeta,i\tau)$, respectively. For definiteness, let us examine the former case considering the latter one in the final result.

Problem (18), (19), (20) in the Time Domain is, of course, not new but it is traditionally analyzed by using Integral Transformation Techniques. The final solution in this situation takes the form either of the time convolution of the Green function and the excitation or inverse transformation form. In the former case, it can be made with help of direct numerical computations, see, e.g. [5]. The classical asymptotic approaches are used in the latter case, see, e.g., [7].

In this paper we use the Separation of Variables Method wherein problem (18), (19), (20) can be more readily solved in the Time Domain. It is a key-point idea of the paper that the separation of variables is possible in many types of space-time coordinate systems which provide a new form of analytical solution to the problem (18), (19), (20) in the Time Domain. A full set of separable coordinate systems for one-dimensional Klein-Gordon equation (18) is listed in [6].

We use the transformations

$$\tau = u\,\mathrm{ch}\,v,\quad \zeta = u\,\mathrm{sh}\,v,\quad 0 \le u < \infty, -\infty < v < \infty \tag{27}$$

for going from original (ζ,τ) to new coordinate system (u,v) with variables defined as follows

$$u = \sqrt{\tau^2 - \zeta^2},\quad v = \frac{1}{2}\ln\left(\frac{\tau+\zeta}{\tau-\zeta}\right),\quad 0 < |\zeta| < \tau < \infty. \tag{28}$$

Parametrization (27) and (28) covers only the sector of the Minkowski (ζ,τ) - plane given by $\tau \pm \zeta > 0$. Similar separable coordinates can be set up in the three other quadrants. In the problem in question, it is assumed that in quadrants different from $\tau \pm \zeta > 0$ one has a zero EM field. Let us apply the separation of (u, v) variables in ordinary way

$$f(u,v) = V(v)U(u) \tag{29}$$

for the rearranged Klein-Gordon equation (18)

$$\frac{\partial^2 f}{\partial u^2} + \frac{1}{u^2}\frac{\partial^2 f}{\partial v^2} + \frac{1}{u}\frac{\partial f}{\partial u} + \tilde{\kappa}^2 f(u,v) = 0 \tag{30}$$

to yield its particular solution

$$f(u,v) = \left(A_\nu \exp(\nu v) + B_\nu \exp(-\nu v)\right) \times \left[C_\nu J_\nu(u) + D_\nu N_\nu(u)\right], \tag{31}$$

where $A_\nu, B_\nu, C_\nu, D_\nu$ are arbitrary constants, J_ν and N_ν are Bessel functions of the first and the second kind, respectively, of order ν; ν is arbitrary separation parameter.

For EM fields to be limited all through the waveguide, we need to avoid the $N_\nu(u)$ term since it builds up without bound with $u \to 0$ $(\tau \to \zeta)$. Thus, let us consider the following particular solution to the Klein-Gordon equation returning to the original variables (τ,ζ)

$$f_k^{\pm}(\zeta,\tau) = C_k^{\pm}\left(\frac{\tau - |\zeta|}{\tau + |\zeta|}\right)^{\frac{k}{2}} J_k\left(\tilde{\kappa}\sqrt{\tau^2 - \zeta^2}\right), \tag{32}$$

$$0 \le |\zeta| < \tau,\quad k = 0,1,2,\dots .$$

The functions $f_k^{\pm}(\zeta,\tau)$ correspond to right- or left-going waves which propagate in the $\pm$ z-direction from the origin $z = 0$, respectively. We form the solution to the Klein-Gordon equation (18) by superposition of all possible propagation waves (32)

$$f^{\pm}(\zeta,\tau) = \sum_{k=0}^{\infty} C_k \left(\frac{\tau - |\zeta|}{\tau + |\zeta|}\right)^{\frac{k}{2}} \times J_k\left(\tilde{\kappa}\sqrt{\tau^2 - \zeta^2}\right),\quad 0 \le |\zeta| < \tau \tag{33}$$

to satisfy boundary condition (19)

$$f^{\pm}(\zeta,\tau)\Big|_{\zeta=0} = e^{\tau\sigma}\varphi(\tau) = \sum_{k=0}^{\infty} C_k J_k\left(\sqrt{1-\sigma^2}\,\tau\right),\quad 0 \le |\zeta| < \tau. \tag{34}$$

In Eqs. (33) and (34) we assume that $0 \le \sigma < 1$. In the $1 < \sigma < \infty$ case, one needs to derive similar equations by substituting Bessel functions for the modificate Bessel functions, vis.,

$$J_k\left(\sqrt{1-\sigma^2}\sqrt{\tau^2-\zeta^2}\right) \to I_k\left(\sqrt{\sigma^2-1}\sqrt{\tau^2-\zeta^2}\right),$$

and desired coefficients C_k for new $C_k \to \tilde{C}_k$.

Expressions like (33) have been already used for the description of transients in waveguides, but they were derived with more complicated and less general methods. Relation (33) may be considered as an expansion of the required solution in *one of the possible evolutionary basic set* for Klein-Gordon equation solutions.

For an arbitrary signal $f(t)$ it is practicable to determine the coefficients C_k in Neumann series (34) using their association with the coefficients in Mackloren series B_n [8], vis.,

$$C_0 = B_0,\quad C_k = k\sum_{n=0}^{\le k/2} 2^{k-2n}\frac{(k-n-1)!}{n!}B_{k-2n}$$

$$B_n = \frac{1}{n!}\left(\frac{d}{d\tau}\right)^n \left(e^{\sigma\tau}\varphi(\tau)\right)\Big|_{\tau=0}\quad (\text{for } 0 \le \sigma < 1). \tag{35}$$

Hence, the expansion coefficients C_k became readily available from the time derivation of an excitation function (or input signal).

5. Propagation of EM step signal

Let us consider a signal generated in a waveguide by the EM step input ($\varphi(\tau)\equiv 1$ in Eq. (15)).

$$F(\zeta,\tau)\big|_{\zeta=0}=H(\tau)=\begin{cases}1, & \tau\geq 0;\\ 0, & \tau<0.\end{cases} \tag{36}$$

The unit step signal is of great interest in view of the following reason. An arbitrary signal (continuous or discontinuous) may be obtained (exactly or approximately) by means of a series expansion in time-shifted step functions or DuHamel's integral, vis.,

$$\varphi(\tau)=\sum_{k=0}^{N}A(k)H(\tau-k\Delta\tau),$$

$$\varphi(\tau)=\int_0^{\tau}A(\tau')H(\tau-\tau')d\tau'.$$

As it have been rightly pointed out in [9], for study of transients of EM waves the step function is of a significance comparable to that of sinusoidal waves for study of the steady state EM waves.

The simplest signals that use the solution to the unit step signal problem are the Walsh functions. Since the Walsh functions are two-valued they can be used in the transmission of information digitally. In signal processing, the Walsh transform proves to be much more easier and faster than the Fourier transform, since its computation involves only additions and subtractions.

Rather than use Eqs. (35) directly to obtain coefficients C_k, we take advantage of the well-known [10] relation

$$\exp\left[\frac{x}{2}\left(\beta-\frac{1}{\beta}\right)\right]=$$

$$=J_0(x)+\sum_{n=1}^{\infty}\left(\beta^n+(-1)^n\beta^{-n}\right)J_n(x),$$

$$\exp\left[\frac{x}{2}\left(\beta+\frac{1}{\beta}\right)\right]=I_0(x)+\sum_{n=1}^{\infty}\left(\beta^n+\beta^{-n}\right)I_n(x)$$

to obtain

$$C_k=\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{k}{2}}+(-1)^k\left(\frac{1-\sigma}{1+\sigma}\right)^{\frac{k}{2}}\quad(\text{for } 0\leq\sigma<1),$$

$$C_k=\left(\frac{1+\sigma}{\sigma-1}\right)^{\frac{k}{2}}+\left(\frac{\sigma-1}{1+\sigma}\right)^{\frac{k}{2}}\quad(\text{for } 1<\sigma<\infty).$$

Thus, the net result of the lossy waveguide problem for the unity step signal is given by

$$F^{\pm}(\zeta,\tau)=\begin{cases}H(\tau-|\zeta|)e^{-\sigma\tau}\Big\{J_0\left(\sqrt{1-\sigma^2}\sqrt{\tau^2-\zeta^2}\right)+ & (for\quad 0\leq\sigma<1)\\ +\sum_{k=1}^{\infty}\left[\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{k}{2}}+(-1)^k\left(\frac{1-\sigma}{1+\sigma}\right)^{\frac{k}{2}}\right]\left(\frac{\tau-|\zeta|}{\tau+|\zeta|}\right)^{\frac{k}{2}}J_k\left(\sqrt{1-\sigma^2}\sqrt{\tau^2-\zeta^2}\right)\Big\}; & (37)\\ H(\tau-|\zeta|)e^{-|\zeta|}\ ; & (for\quad\sigma=1)\quad(38)\\ H(\tau-|\zeta|)e^{-\sigma\tau}\Big\{I_0\left(\sqrt{\sigma^2-1}\sqrt{\tau^2-\zeta^2}\right)+ & (for\quad 1<\sigma<\infty)\\ +\sum_{k=1}^{\infty}\left[\left(\frac{1+\sigma}{\sigma-1}\right)^{\frac{k}{2}}+\left(\frac{\sigma-1}{1+\sigma}\right)^{\frac{k}{2}}\right]\left(\frac{\tau-|\zeta|}{\tau+|\zeta|}\right)^{\frac{k}{2}}I_k\left(\sqrt{\sigma^2-1}\sqrt{\tau^2-\zeta^2}\right)\Big\}. & (39)\end{cases}$$

6. Numerical / Graphical Results and Discussion

The injection of ohmic losses in a waveguide medium leads to the change of size, shape and time of the *transients* but has no effect on the form of a *steady-state* process. The last one is derived from solution (37), (38), (39) if time τ tends to infinity and is nonzero for the step signal.

Depending on losses value σ, time-dependent transients at a fixed distance ζ = *const* may be classed (see Fig. 1) into oscillated function form ($0\leq\sigma<1$, Eq.(37)) or monotonically increased function form ($1\leq\sigma<\infty$, Eq.(39)). The case $\sigma=1$, Eq.(38), corresponds to the absence of transients. The time-constant solution in it just yields the shape of a steady-state process

$$\lim_{\substack{\tau\to\infty \\ 0\le\sigma<\infty}} F(\tau,\zeta,\sigma)\to\exp(-\zeta). \qquad (40)$$

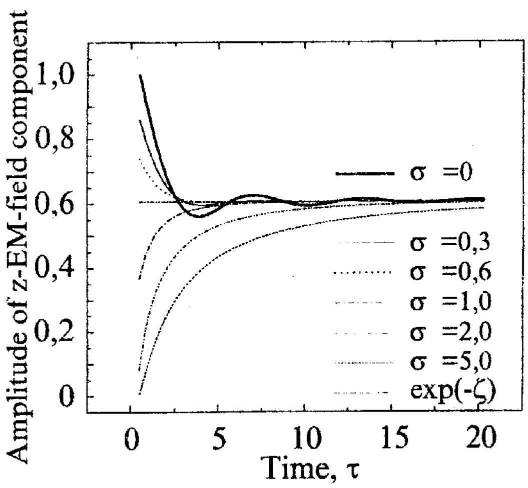

Fig. 1. *Time-dependent signal EM-field at ζ=1 for different conductivities (losses)*

Increasing the conductivity σ (i.e. losses) leads to decreasing the transient time in case of $0 \le \sigma < 1$ and to increasing it in case of $1 < \sigma < \infty$.

In case of $0 \le \sigma < 1$, the more oscillations occur with distance from the excitation and less oscillations occur with increasing the conductivity σ (i.e. losses). The latter conclusion is illustrated, for example, by the width of a front spike that can be estimated as

$$\Delta \approx \zeta - \sqrt{\zeta^2 + \frac{j_{01}}{1-\sigma^2}}\,, \qquad (41)$$

where j_{01} =2,44 is the first zero of Bessel function $J_0(x)$.

For all values of conductivity σ (i.e. losses), it is true that a signal front propagates with the velocity of light in the medium $v_l = 1/\sqrt{\varepsilon_0\mu_0\varepsilon\mu} = c/\sqrt{\varepsilon\mu}$. Front remains its true value for lossless medium and therefore remains always well-marked. To detect velocity of the signal front in a waveguide with arbitrary conductivity (i.e. losses), we need however to assume a technically perfect threshold detector with an infinite resolution because both the signal front and its "tail" decay exponentially on passage through the lossy medium.

The most interesting is the case of lossless waveguide filling which is derived by substitution $\sigma = 0$ in Eq.(37). The absence of power factor from zero term of this solution is of importance in estimation near the signal front when (τ - ζ) is small. This is most pronounced far from the excitation point when the contribution from each wave component $F_k^{\pm}(\zeta,\tau)$, $k > 0$ is negligible in comparison with $F_0^{\pm}(\zeta,\tau)$ (see Fig. 2).

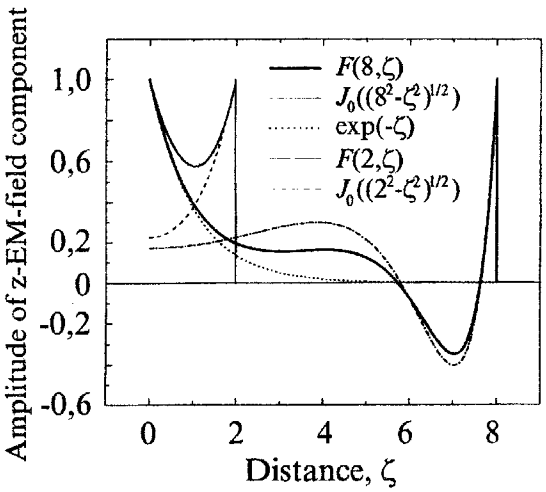

Fig. 2. *Comparison between spartial distribution of the signal EM-field and its approximation at τ=2 and τ=8*

Thus, the more is time of signal propagation in waveguide, the more is distance from the signal front when spatial distribution of the signal is characterized by

$$F^{\pm}(\zeta,\tau)\Big|_{\substack{(\tau-\zeta)\to 0\\ \zeta\to\infty}} \cong F_0^{\pm}(\zeta,\tau) =$$

$$= H(\tau - |\zeta|)\,J_0\left(\sqrt{\tau^2-\zeta^2}\right). \qquad (42)$$

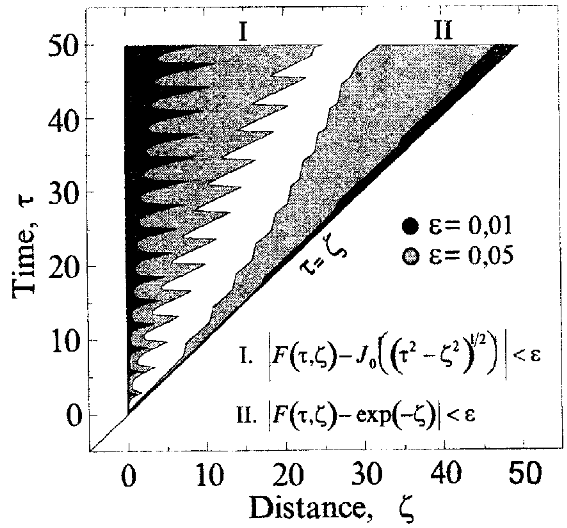

Fig. 3. *The space-time domains in which approximations are valid*

Quantitative analysis of (ζ,τ) - domain on which approximate evaluations (40), (42) are applicable is depicted in Fig. 3. Every point of the painted domain yields that distance ζ and that instant τ for which the difference $\left|F^{\pm}(\tau,\zeta) - J_0\left(\sqrt{\tau^2-\zeta^2}\right)\right|$ or $\left|F^{\pm}(\tau,\zeta) - \exp(-\zeta)\right|$ is less then a given error ε. Physically, ε may be considered as resolving power of threshold detector measured the EM field amplitude.

The plot of time-dependent energy $W(\zeta,\tau)$ divided on the $\kappa^2\varepsilon\varepsilon_0$ for E-waves or $\kappa^2\mu\mu_0$ for H-waves and normalized energy transport velocity $v_S(\zeta,\tau)/v_l$ given by (25) are depicted in Fig. 4.

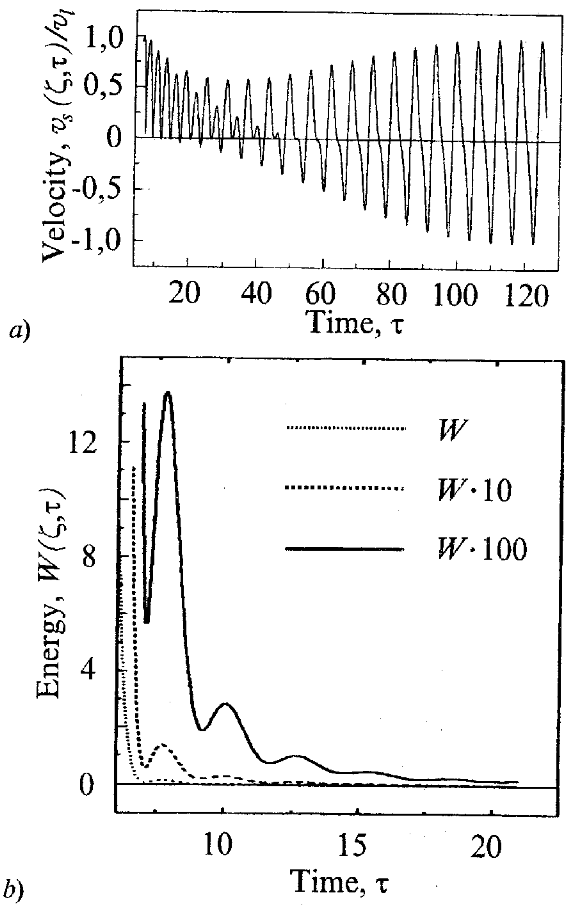

Fig. 4. *Time dependent energy $W(\zeta,\tau)$ and normalized energy transport velocity $v_s(\zeta,\tau)/v_l$ at ζ=6*

Simulation of energy transport with a computer gives following results:

- The energy of step signal is basically concentrated near the signal front on which it has its maximum. Hence the transport of the greatest part of energy and the signal propagation are in the same direction.
- Energy of the transients distant from the signal front is two-directed and vanishingly small.
- With away from excitation point a waveguide impart largely the guided properties to the energy of the signal.
- Inclusion of ohmic losses to the waveguide medium leads not only to the predicted decreasing of signal energy but also to its further concentration in vicinity of signal front. Differences between the energetical characteristics of E-waves and H-waves are not significant.

References

1. S. L. Dvorak, and D.G. Dudley. IEEE Transactions on Electromagnetic Compatibility. 1995, **37**, No. 2, p. 192-200.
2. V.V. Borisov. Transients in Waveguides (in Russian). Publishing House of Leningrad State University, Leningrad, 1991, 156 p.
3. O.A. Tretyakov. Essentials of Nonstationary and Nonlinear Electromagnetic Field Theory. In: Analytical and numerical methods in electromagnetic wave theory, Ed. M. Ideman, M. Hashimoto, and O.A. Tretyakov. Science House, Tokyo, Japan, 1993, p. 123-146.
4. O.A. Tretyakov. Proc. IEEE Antennas and Propagation Society International Symposium. Seattle, 1994, **3**, p. 1973-1976.
5. G. J. Kristensson. Electromagn. Wave and Appl. 1995, **9**, No. 5/6, p. 645-671.
6. W. Miller. Symmetry and Separation of Variables. Addison-Wesley Publishing Company, London, 1977, 342 p.
7. A.E. Karbowiak. Proc. IEE (London), part C. 1957, **104**, p. 339-348.
8. G.N. Watson. A treatise on the theory of Bessel functions. Cambridge, Cambridge Univ. Press, 1958.
9. H.F. Harmuth. Propagation of Nonsinusoidal Electromagnetic Waves. London, Academy Press Inc., 1986, 240 p.
10. A.P. Prudnikov, Y.A. Brychkov, and O.I. Marichev. Integrals and Series, volume 2: Special Functions. Gordon and Breach Science Publishers, New York, 1986, 752 p.

Распространение электромагнитного ступенчатого сигнала в волноводе с потерями

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Рассматривается распространение нестационарного электромагнитного поля в волноводе произвольного поперечного сечения с идеально проводящей поверхностью, заполненном средой с потерями. Используются волноводные эволюционные уравнения, сводящие во временной области исходную начально-краевую векторную задачу для уравнений Максвелла к интегрированию одномерного скалярного уравнения Клейна-

Гордона. Решение находится в виде разложения по подходящему нестационарному базису решений данного уравнения. Коэффициенты разложения легко определяются из аналогичного представления входного сигнала. Приводятся аналитические формы ранне- и поздне- временного приближений распространяющегося сигнала с условиями их применимости по отношению к точному решению.

Поширення електромагнітного ступінчастого сигналу в хвилеводі з втратами

К.М. Ємельянов, С.Б. Нікітський, О.О. Третьяков

Розглядається поширення нестаціонарного електромагнітного поля в хвилеводі довільного поперечного перерізу з ідеально провідною поверхнею, що заповнений середовищем з втратами. Використовуються хвилеводні еволюційні рівняння, що зводять в області часу вихідну початково-крайову векторну задачу для рівнянь Максвела до інтегрування одномірного скалярного рівняння Клейна-Гордона. Розв'язок знаходиться у вигляді розкладу по відповідному нестаціонарному базису рішень даного рівняння. Коефіцієнти розкладу легко визначаються з аналогічного представлення вхідного сигналу. Приводяться аналітичні форми рано- та пізно- часового наближень сигналу, що розповсюджується, з умовами їх застосування по відношенню до точного розв'язку.