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Interpolation Method for Evaluation of Periodic Green Function in Problems of Diffraction

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The algorithm is proposed to calculate the Green function for the problem of the diffraction of the plane electromagnetic wave on infinite periodical grating.

Investigation of properties of fields, diffracted by a system of scatterers, is an important problem of current radiophysics. Description of such phenomena in resonance frequency range, where length of the wave is proportional to the scatterers dimensions, is possible only with a rigorous solution of corresponding diffraction problem. An infinite periodic grating may serve as one of the models of scatterers system, which has resonance features. Papers of a number of authors, particularly [1- 5], are devoted to investigation of such structures. It was shown, that the significal difficulties arise while the scattered field is calculated in the case of an arbitrary (noncanonical) scatterers geometry. One of the ways of the efficiency increasing of such calculations is proposed below.

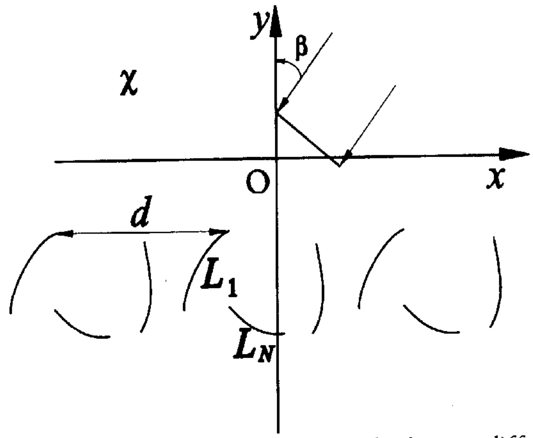

Fig.1. *Plane wave radiation of multielement diffraction grating.*

Consider multielement d-periodic grating in homogeneous isotropic media with wave number χ. One (curvilinear) period of the grating contains N cylindrical perfectly conducting screens with generatrices parallel to Oz axis. Cross-sections of the screens by xOy plane are open smooth Lyapunov-type contours L_k, $k = \overline{1,N}$ (Fig. 1). The grating is irradiated by a unit-amplitude plane electromagnetic wave with time dependence $\exp[-i\omega t]$ (ω- circular frequency). Let β is angle of wave incidence to Oy axis. Then such problem is reduced to Helmholtz equation solution on the grating period, which satisfies the following conditions: of Dirichlet (E-polarization) or Neumann (H-polarization) on the arcs L_k, $k = \overline{1,N}$, of Meixner-type near the screens ribs (L_k are end-points). If $|y| \to \infty$, the scattered field should not contain waves propagating from infinity.

Green Function of the Problem

The method of integral equations, which contain periodic Green function in their kernels, may be one of the methods of such problem solution [1-3,6,7]. In the E-case it may be written as [2,6]

$$G(t,z) = \frac{i}{4} H_0^{(1)}(\chi r) +$$
$$+ \frac{i}{4} \sum_{\substack{u=-\infty \\ u\neq 0}}^{+\infty} H_0^{(1)}(\chi |q+ud|) \exp[i\chi du \sin\beta],$$

$$q=t-z,\ r=|q|, \tag{1}$$

where $H_0^{(1)}(z)$ is a Hankel function of the first kind with a logarithmic singularity at $z=0$; t is a point of $L = \cup L_k$ contour; $z=x+iy$ is a complex coordinate of a view point; $u = -\infty,+\infty$ is a period number.

To avoid the convolution of the series (1) (which have a weak convergence [3,7]) and to provide correctness in mathematical transformations, it has been proposed in [5, 8] to utilize a known integral presentation of Hankel function.

$$\frac{i}{4} H_0^{(1)}(\chi r) = \frac{1}{4\pi} \int_\Gamma \frac{1}{\nu} e^{-|\Im\{q\}|\nu} e^{i\xi\Re\{q\}} d\xi,$$

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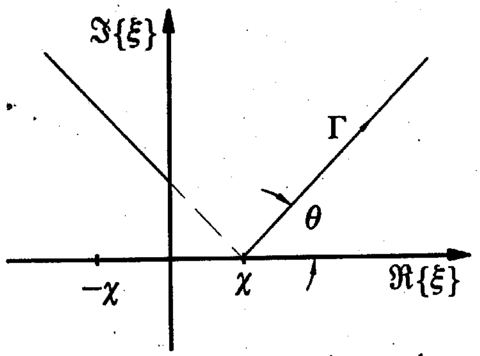

Fig.2 Integration contour in complex x-plane.

$$\nu = \sqrt{\xi^2 - \chi^2},\ \Re\{\nu\} \ge 0. \qquad (2)$$

The condition $\Re\{\nu\}\ge 0$ determines the Riemann's surface to satisfy the conditions at the infinity. If Green function is calculated in the near zone, the integration contour Γ may be chosen according to Fig. 2. The dotted line shows that the contour passes through another Riemann's surface, for which

$$\Im\{\nu\} < 0,\ \Im\{\xi\} \ge 0,\ 0 \le \Re\{\xi\} \le \chi,\ \xi \in \Gamma. \qquad (3)$$

Angle θ has been determined from the condition $tg(\theta)=\Re\{\theta\}/|\Im\{\theta\}|$, which (together with (3)) removes oscillations and guarantees the fastest vanishing of integrand function in presentation (2). Then Green function looks like

$$G(t,z) = \frac{i}{4} H_0^{(1)}(\chi r) + \sum_{u=1}^{+\infty} \frac{1}{4\pi} \exp[-i\chi d u \sin\beta] \int_\Gamma \frac{1}{\nu} e^{-|\Im\{q\}|\nu} e^{-i\xi\Re\{q-ud\}} d\xi + \sum_{u=1}^{+\infty} \frac{1}{4\pi} \exp[i\chi d u \sin\beta] \int_\Gamma \frac{1}{\nu} e^{-|\Im\{q\}|\nu} e^{i\xi\Re\{q+ud\}} d\xi \qquad (4)$$

It is possible now to insert the sum under the integral, to use formula for geometric progression sum, and to obtain the following expression:

$$G(t,z) = \frac{i}{4} H_0^{(1)}(\chi r) + S_+(t,z,d,\beta) + S_-(t,z,d,\beta);$$

$$S_+(t,z,d,\beta) = e^{-i\chi d \sin\beta} \frac{1}{4\pi} \int_\Gamma \frac{e^{-|\Im\{q\}|\nu - i\xi\Re\{q-d\}}}{\nu\left(1 - e^{id(\xi - \chi\sin\beta)}\right)} d\xi,$$

$$S_-(t,z,d,\beta) = e^{i\chi d \sin\beta} \frac{1}{4\pi} \int_\Gamma \frac{e^{-|\Im\{q\}|\nu + i\xi\Re\{q+d\}}}{\nu\left(1 - e^{id(\xi + \chi\sin\beta)}\right)} d\xi. \qquad (5)$$

The case of the normal incidence of plane wave (b=0) leads to

$$S_+(t,z,d,0) = S(d-r,d);$$
$$S_-(t,z,d,0) = S(d+r,d);$$

$$S(z,d) = \frac{1}{4\pi} \int_\Gamma \frac{e^{-|y|\nu + i\xi x}}{\nu\left(1 - e^{id\xi}\right)} d\xi. \qquad (6)$$

It should be mentioned, that integration contour, drawn according to Fig. 2, is easy in numerical realization, allows to transform presentation (4) into (5), but fails to calculate the Green function in the far zone ($\mathrm{Im}\{z\}\to\pm\infty$), because, if

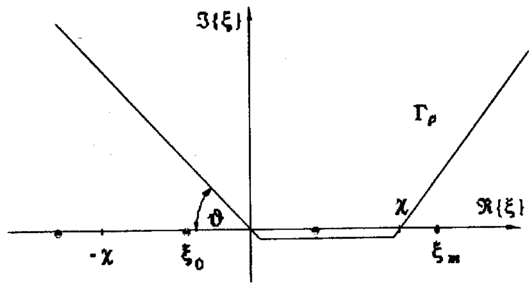

Fig. 3. *Account for existing of poles* ξ_m.

$|y|\to\infty$, the integral is diverging on the dotted part of the integration contour. Consider another method of the integration contour tracing (Fig. 3) according to which poles, situated on the real axes of complex plane ξ, remain on the left hand. Such poles should be taken into account by formula

$$\int_\Gamma \frac{e^{-|\Im\{q\}|\nu \mp i\xi\Re\{q\mp d\}}}{\nu\left(1 - e^{id(\xi\mp\chi\sin\beta)}\right)} d\xi = \int_{\Gamma_p} \frac{e^{-|\Im\{q\}|\nu \mp i\xi\Re\{q\mp d\}}}{\nu\left(1 - e^{id(\xi\mp\chi\sin\beta)}\right)} d\xi - \frac{2\pi}{d} \sum_{m=1}^{n_\pm} \frac{e^{|\Im\{q\}|\nu_m \mp i\xi_m\Re\{q\mp d\}}}{\nu_m},$$

$$\nu_m = \sqrt{\xi_m^2 - \chi^2},\ \mathrm{Re}\{z\}\ge 0. \qquad (7)$$

Here $n_\pm = \left[d\chi(1 \mp \sin\beta)/(2\pi)\right]$ is a number of the poles, in which residues are calculated; coordinates of the poles' distribution are given by formula

$$\xi_m = 2\pi m/d \pm \chi \sin\beta, \quad m = \overline{1, n_\pm} \qquad (8)$$

Then expressions for S_+ and S_- in formula (5) will be

$$S_\pm(t,z,d,\beta) = e^{\mp i\chi d \sin\beta} \frac{1}{4\pi}\left[\int_{\Gamma_p} \frac{e^{-|\Im\{q\}|\nu \mp i\xi\Re\{q\mp d\}}}{\nu\left(1-e^{id(\xi\mp\chi\sin\beta)}\right)}d\xi - \frac{2\pi}{d}\sum_{m=1}^{n_\pm}\frac{e^{-|\Im\{q\}|\nu_m \mp i\xi_m\Re\{q\mp d\}}}{\nu_m}\right]. \qquad (9)$$

Note, that if $\beta=0$, the pole ξ_0 will appear on the integration contour Γ_p at point (0,0). In this case the integral in formula (6) will be

$$\int_\Gamma \frac{e^{-|y|\nu + i\xi x}}{\nu\left(1-e^{id\xi}\right)}d\xi = \int_{\Gamma_p} \frac{e^{-|y|\nu+i\xi x}}{\nu\left(1-e^{id\xi}\right)}d\xi - \pi i \left.\frac{e^{-|y|\nu+i\xi x}}{\nu\left(1-e^{id\xi}\right)'}\right|_{\xi=0} - 2\pi i \sum_{m=1}^{n}\left.\frac{e^{-|y|\nu+i\xi x}}{\nu\left(1-e^{id\xi}\right)'}\right|_{\xi=\xi_m} =$$

$$= \int_{\Gamma_p} \frac{e^{-|y|\nu+i\xi x}}{\nu\left(1-e^{id\xi}\right)}d\xi + \frac{\pi i}{\chi d}e^{i\chi|y|} + \frac{2\pi}{d}\sum_{m=1}^{n}\frac{e^{|y|\nu_m + ix\xi_m}}{\nu_m}, \qquad (10)$$

were $n=[\chi d/2\pi]$; $\xi_m = 2\pi m/d$, $m = \overline{1,n}$. Such contour modification is valid for the far zone and, besides, allows to provide an accurate Green's function calculation on the $L=UL_k$ contours. Taking into account, that integrals in (9) and (10) are vanishing at the infinity, and providing some simplifications, if $|y|\to\infty$ (far zone) it is possible to write

$$G^\infty(t,z,d,\beta) = \frac{i}{2\chi d}\sum_{m=-n_-}^{n_+}\frac{e^{i\chi\left(\cos^*(\beta,m)|\Im\{q\}| - \sin^*(\beta,m)\Re\{q\}\right)}}{\cos^*(\beta,m)}, \qquad (11)$$

$$\cos^*(\beta,m) = \sqrt{\cos^2\beta - 2\frac{2\pi m}{\chi d}\sin\beta - \left(\frac{2\pi m}{\chi d}\right)^2},$$

$$\sin^*(\beta,m) = \sin\beta + \frac{2\pi m}{\chi d}.$$

To compare the accuracy of calculations, performed by the presented methods, Hankel function was calculated by formula (2) with the integration contour, drawn as in Figs. 2 and 3. The results were compared with the exact ones given in literature. Preassigned accuracy $\varepsilon=10^{-6}$ was obtained for all testing points from the area of parameters' variation: $0.125\lambda \le x \le 1.185\lambda$; $0 \le y \le 1.5\lambda$.

The following algorithm was used for Zommerfeld-type integrals (6) calculation. The integration contour in Fig. 2. was divided into four parts: two finite segments, which have c point as one of their ends, and two semi-infinite, which complete the finite segments to the Γ contour. Substitution $\xi=\chi+\xi'(t-1)$ was introduced, were parameter ξ determines the integration direction, $0\le t\le 1$. Behaviour of expression ξ/ν if $\xi\to\chi$ is easy to establish from

$$-i\frac{\sqrt{1-\tau^2}}{\sqrt{\xi^2-\chi 2}} = \frac{\sqrt{1-\tau^2}}{\sqrt{\chi^2-(\chi-\xi'+\xi'\tau)^2}} = \frac{\sqrt{1+\tau}}{\sqrt{\xi'}\sqrt{\chi+(\chi-\xi'+\xi'\tau)}}.$$

Thus, the equality takes place

$$\lim_{\tau\to 1}\frac{\sqrt{1+\tau}}{\sqrt{\xi'}\sqrt{\chi+(\chi-\xi'+\xi'\tau)}} = \frac{1}{\sqrt{\xi'\chi}}.$$

Then for an arbitrary function $\varphi(\xi)$, analytical in the complex plane ξ, such equations are valid:

$$\int_{\Gamma_s}\frac{\varphi(\xi)}{\nu}\exp[-\nu\alpha + i\xi\beta]d\xi = $$

$$= \int_0^1\frac{\varphi(\xi)}{\nu}\exp[-\nu\alpha + i\xi\beta]\xi' d\tau =$$

$$= \xi'\left[\int_0^1\frac{1}{\nu}\left\{\varphi(\xi)\exp[-\nu\alpha + i\xi\beta] - \frac{\xi}{\chi}\varphi(\chi)\exp[i\chi\beta]\right\}d\tau\right] + \frac{\varphi(\chi)}{\chi}\exp[i\chi\beta]\int_{\Gamma_s}\frac{\xi}{\nu}d\xi =$$

$$= \xi' \left[\int_0^1 \frac{1}{\nu} \left\{ \varphi(\xi) \exp[-\nu\alpha + i\xi\beta] - \frac{\xi}{\chi} \varphi(\chi) \exp[i\chi\beta] \right\} d\tau \right] - \frac{\varphi(\chi)\exp[i\chi\beta]}{\chi} \nu \Big|_{\tau=0}. \quad (12)$$

It is easy to calculate the obtained right part of formula (12), due to it regularity, with the help of quadratures. By putting: Γ_s - the finite segment of Γ contour; $\alpha=\Im\{z\}$, $\beta=\Re\{z\}$; $\varphi(\xi)=(1-\exp\{i\xi d\})^{-1}$, we obtain the formula for integral (6) evaluation.

Interpolation Polynomial

For numerical solution of a diffraction problem it is necessary to perform Green function's calculation in a great number of points. To decrease the time of calculation of integrals of type (5,6) we construct a three-dimensional interpolation polynomial. It is known, that interpolation Lagrange polynomial on Tchebyshev's nodes $\tau=\xi_k=\cos\theta_k$ ($\theta_k=(2k-1)\pi/2n$, $k=\overline{1,n}$; n is the number of nodes) can be written in the form of [9]

$$\varphi(\tau) \approx -\frac{1}{n}\sum_{k=1}^{n}\varphi(\xi_k)\frac{T_{n-1}(\xi_k)T_n(\tau)}{\xi_k-\tau}. \quad (13)$$

Such polynomial interpolates approximately an arbitrary function $\varphi(t)$. Equality (13) is an accurate one for the case, when $\varphi(t)$ is a polynomial of order not higher than n-1. Indefinite form in the case of matching of the interpolation point with a node x_k is exposured using Lopitall's rule and taking into account Chebyshev's first type polynom definition:

$T_n(t)=\cos[n\arccos t]$,

$$\left.\frac{T_n(\tau)}{\xi_k-\tau}\right|_{\tau=\xi_k} = \mp(-1)^k\frac{n}{\sin\xi_k}.$$

Acting likewise for the case of three variables: $x=\Re\{z\}$; $y=\Im\{z\}$; d, and supposing, that for two arbitrary arguments the function $S(x,y,d)$ is approximated by a polynomial with respect to the third one, by an analogy to [10], we obtain

$$S(x,y,d) \approx -\frac{T_{m_x}(x)T_{m_y}(y)T_{m_d}(d)}{m_x m_y m_d}\sum_{k_x=1}^{m_x}\sum_{k_y=1}^{m_y}\sum_{k_d=1}^{m_d}\frac{T_{m_x-1}(\xi_{k_x})}{\xi_{k_x}-x}\frac{T_{m_y-1}(\xi_{k_y})}{\xi_{k_y}-y}\frac{T_{m_d-1}(\xi_{k_d})}{\xi_{k_d}-d}S(\xi_{k_x},\xi_{k_y},\xi_{k_d}). \quad (14)$$

To increase the efficiency of calculation by formula (14), the ranges of value of each variable may be divided into smaller parts, what gives a possibility to obtain the necessary accuracy without enlarging of values of the parameters m_x, m_y, m_d.

For the presented algorithm testing, the integral (10) was directly calculated at 2000 arbitrary choosed points. After filling in the array $S(\xi_{k_x},\xi_{k_y},\xi_{k_d})$, the interpolation was performed in the same points. The absolute accuracy of interpolation $e=10^{-5}$ was achieved with the following parameters of interpolation polynomial: $m_x=m_y=m_d=5$; number of parts for wave length for appropriate arguments: $l_x=l_y=l_d=8$. Maximum relative error has not exceeded 0.01%, and - 0.001% at the majority points. The time of integral (10) calculation at 1000 points was 2 minutes 1.88 seconds; the time of its calculation using the interpolation procedure is 4.89 seconds. Particular case for d=0.51 $\lambda=\pi$ and some $z=x+iy$ is presented in the Table. With significant decrease of evaluation time, the achieved relative accuracy of calculation using the interpolation polynomial, is commensurable with the accuracy of direct calculation of integral (10). It gives the ground to confirm the high efficiency of the used approach. Note, that Green function contains the endless sum of Hankel functions, so, if $z-t_k+d=0$ the addends $S_\pm(t_k,z,d,\beta)$ may admit stationary logarithmic singularities. For their evaluation in such cases the next expressions were used

$$S_\pm(t,z,d,\beta) =$$

$$= e^{\mp i\chi d\sin\beta}\left(\frac{i}{4}H_0^{(1)}(\chi|d\mp d|)+S_\pm(t\mp d,z,d,\beta)\right) \quad (15)$$

Thus, the proposed algorithm allows to calculate the diffracted field in the near zone, as in the far one. Besides, the interpolation polynomial utilization essentially decreases the evaluation time preserving the accuracy, what essentially increases the efficiency of the algorithm utilization, and in the case of scatterer in the form of diffrac tion grating gives a possibility to solve diffraction problems in the acceptable time.

Table

Coordinates		Integral's value: calculated / interpolated		Interpolation error: *absolute* / *realtive(%)*	
y	x	$\Re$	$\Im$	$\Re$	$\Im$
0.000	3.142	-.314106E+00 -.314109E+00	-.285398E+00 -.285402E+00	-.301003E-05 0.0011	-.378489E-05 0.0002
	4.398	.146192E+00 .146193E+00	-.312767E+00 -.312772E+00	.146031E-05 0.0013	-.414252E-05 0.0001
	5.655	.294246E+00 .294249E+00	.387681E-01 .387678E-01	.238419E-05 0.0008	-.294298E-06 0.0002
	6.912	.467053E-01 .467053E-01	.259454E+00 .259452E+00	-.357628E-05 0.0009	-.169873E-05 0.0012
	8.168	-.211233E+00 -.211225E+00	.112342E+00 .112321E+00	.813603E-05 0.0072	-.206977E-04 0.0060
-6.283	3.142	-.179931E+00 -.179932E+00	.326995E+00 .326998E+00	-.183284E-05 0.0010	.324845E-05 0.0000
	4.398	-.272924E+00 -.272928E+00	.147816E+00 .147818E+00	-.336766E-05 0.0012	.146031E-05 0.0001
	5.655	-.263374E+00 -.263377E+00	-.558049E-01 -.558059E-01	-.241399E-05 0.0009	-.101700E-05 0.0002
	6.912	-.124999E+00 -.124999E+00	-.206255E+00 -.206258E+00	-.402331E-06 0.0010	-.268221E-05 0.0004
	8.168	.781408E-01 .781455E-01	-.206672E+00 -.206671E+00	.465661E-05 0.0004	.640750E-06 0.0021

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Интерполяционный метод оценки периодической функции Грина в задачах дифракции

З. Назарчук, О. Овсянников, Т. Сеник

Предложен алгоритм для расчета функции Грина для задачи дифракции плоской электромагнитной волны на бесконечной периодической решетке.

Інтерполяційний метод оцінки періодичної функції Гріна в задачах дифракції

З. Назарчук, О. Овсянников, Т. Сенік

Запропоновано алгоритм для обчислення функції Гріна для задачі дифракції плоскої електромагнітної хвилі на нескінченній періодичній гратці.