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Can Physics Be Constructed Axiomatically?

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The number of essences
should not be increased
beyond necessity
(after W. Ockham)

Attempt is made to build up the physics axiomatically. A number of known facts established by experience is taken for granted. The systems under examination are considered as Hamiltonian after that the equations of classical and quantum mechanics and electrodynamics are deduced without supplementary assumptions.

The ancient Greeks called mathematics the Queen of sciences because of the accuracy and logic perfection of mathematical theories and beautiful elegance of the final conclusions.

The typical structure of a mathematical theory is as follows. The foundation is formed by a set of axioms, either related to the real world or appearing as a product of pure thought. The number of axioms in the set necessary and sufficient for constructing the theory is determined by Goedel's theorem [1]. The rest is more development of the first principles. Each step is strictly controlled, every new assertion is proven in a way excluding any logic deficiency. This is what is termed mathematical rigour. Introducing new postulates during the game is strictly forbidden.

Theories of the kind are remarkably beautiful, however the very possibility of their development is conditioned by the simplicity (one should rather say primitive simplicity) of the object. It is essential that the initial assumptions, i. e. the set of axioms, are taken for granted. They are absolute truth subject to no discussions.

In a sense, mathematical theories are standards to be referred to for the construction of other sciences or, speaking of lofty style, they are the ideal. It cannot be reached in a theoretical description of objects of greater complexity but, as of any ideal, its value is in indicating the goal to strive for.

After mathematics, the second simplest science is physics dealing with inanimate matter. While certainly being simpler than, for instance, biology or physiology, it greatly exceeds mathematics in complexity as it studies real rather than ideal object. The knowledge on those objects is limited in view of the limitations of experiment. The situation is further aggravated by the fact that, instead of studying the object in its integrity, the physicist concentrates on a particular manifestation. The physical problem actually consist of two parts, i. e. of collecting the necessary information and abstracting from the irrelevant. The art of the researcher is to separate one from the other.

Application of the mathematical formalism becomes possible through substantially simplified description of the object, since mathematics is suited to represent relatively simple processes. This simplification is the philosophy of theoretical physics, practically implemented in the form of small parameter methods. The value of theoretical results is tested by their capacity for predicting new effects that can be detected in experiment. Thus, the process conforms to the scheme as follows.

1. Experimental result become a foundation for a theoretical description of a physical phenomenon.
2. The theory developed is used to explain known but poorly understood effects and to predict new ones.
3. The predicted effects are discovered in experiment.
4. The process is continued until the theory fails, after which
5. A new set of experimental data is collected, with the aim of either extending the validity of existing theories, or developing new ones to include the former as particular or limiting cases.

Then everything is started again from point 1 on.

The natural question is whether physics can be constructed in a way similar to mathematics, with at least partial use of the scheme 1 through 5. The part of basic axioms apparently should be played by a set of experimental data represented by some equations, which give the way to implementing article 3 of the above program. At this point it would be desirable to have some considerations for indicating the minimum number of experiments to allow constructing a theory of greatest generality. In principle, it should be so general and complete as to spare us of articles 4 and 5. An analog of Goedel's theorem would have been useful, however it is obviously impossible in physics. Perhaps, it can be substituted by Ockham's "razor blade criterion", i. e. the statement formulated as the epigraph to this article.

To the best of the author's knowledge, the only attempt of axiomatic construction of physics has been undertaken by L. Landau and E. Lifschitz in their famous Course of Theoretical Physics [2,3]. The very fact of propounding the problem proved to be a serious achievement, however Landau and Lifschitz failed to fulfil the program.

The reasons have been many, the main apparently being an unfortunate choice of the set of postulates, lacking a direct relation to experiment. Besides, the program was strictly adhered to only in the first two books of the Course, i. e. Mechanics and Field Theory.

The principal postulate of Landau and Lifschitz was the least action principle. However, it does not follow directly from experiment and the values involved in the formulation, namely action and Lagrange's function, have no immediate physical sense, and hence cannot be measured.

The present paper is another attempt of constructing physics in an axiomatic way.

We will consider Hamiltonian systems.

A system is called Hamiltonian if it can be described in terms of two N-dimensional vectors, $\vec{p}$ and $\vec{q}$ in the Euclidean space, depending on the parameter t. The vector $\vec{p}(t)$ is known as momentum, $\vec{q}(t)$ is the coordinate, and the parameter t is called time. Time obeys the condition

$$dt>0, \tag{1}$$

i. e. time can only increase. This is the principle of causality. A dynamic system represented by time and coordinates alone will be called a vacuum, while a system determined by 3-D momentum and coordinate vectors and time is a particle.

The equations governing the time dependence of coordinates and momenta are equations of motion, or evolutionary equations.

They follow from the law

$$\frac{dH(\vec{p},\vec{q},t)}{dt}=\frac{\partial H(\vec{p},\vec{q},t)}{\partial t} \tag{2}$$

where $H(\vec{p},\vec{q},t)$ is known as the Hamilton function. It is the dynamic system's energy represented in terms of coordinates and momenta. Eq. (2) represents the energy conservation law that can be formulated as follows. The energy of a system varies with time if the system's Hamiltonian function is time dependent explicitly. Otherwise, the energy is conserved. Indeed $\frac{\partial H}{\partial t}=0$ infers

$$H(\vec{p},\vec{q})=const. \tag{3}$$

To derive the evolution equations from Eq. (2), recall that the full derivative of H is

$$\frac{dH}{dt}=\frac{\partial H}{\partial \vec{p}}\frac{d\vec{p}}{dt}+\frac{\partial H}{\partial \vec{q}}\frac{d\vec{q}}{dt}+\frac{\partial H}{\partial t},$$

whence

$$\frac{\partial H}{\partial \vec{p}}\frac{d\vec{p}}{dt}+\frac{\partial H}{\partial \vec{q}}\frac{d\vec{q}}{dt}\equiv 0.$$

This identity holds if $\vec{p}(t)$ and $\vec{q}(t)$ obey either of the two equation sets,

$$\frac{d\vec{p}}{dt}=-\frac{\partial H}{\partial \vec{q}},\quad \frac{d\vec{q}}{dt}=\frac{\partial H}{\partial \vec{p}} \tag{4}$$

or

$$\frac{d\vec{p}}{dt}=\frac{\partial H}{\partial \vec{q}},\quad \frac{d\vec{q}}{dt}=-\frac{\partial H}{\partial \vec{p}} \tag{5}$$

Eqs. (4) and (5) are equivalent and can be transformed one to the other by simple substitutions , e. g. $\vec{q}\to-\vec{q}$ and $\vec{p}\to\vec{p}$. They are known as Hamilton's equations. In what follows, we will use Eq. (4). It can be further transformed in the following way. Let us go from the Lagrange variables in which terms Eq. (4) is written, over to Euler's. Then $\vec{p}$ will be a function of both $\vec{q}$ and t. The first equation in the set takes the form (with account of the second)

$$\frac{\partial \vec{p}}{\partial t}+\frac{\partial p_i}{\partial \vec{q}}\frac{\partial H}{\partial p_i}=-\frac{\partial H}{\partial \vec{q}}, \tag{6}$$

$$\frac{\partial \vec{p}}{\partial t}=-\nabla_{\vec{q}}H, \tag{7}$$

where $\nabla_{\vec{q}}$ denotes the full derivative with respect to $\vec{q}$. It is evident from Eq. (7) that $\vec{p}$ is a potential function. Let the respective potential be S,

$$\vec{p}=\nabla_{\vec{q}}(S). \tag{8}$$

Then it obeys the equation

$$\frac{\partial S}{\partial t}+H(\nabla_{\vec{q}}S,\vec{q},t)=0. \tag{9}$$

In mechanics S bears the name of action and Eq. (9) is known as the Hamilton-Jacobi equation. It is a different form of Hamilton's equation. Similarly, one can obtain

$$\frac{\partial \sigma}{\partial t}-H(\vec{p},\nabla_{\vec{p}}\sigma,t)=0, \tag{10}$$

in which equation the action σ is regarded as a function of $\vec{p}$ and t, and $\vec{q}$ is defined as

$$\vec{q}=\nabla_{\vec{p}}\sigma. \tag{11}$$

Eq. (4) implies the conservation laws as follows.

If H is independent of $\vec{q}$, then $\vec{p}=const$, while an H independent of $\vec{p}$ implies $\vec{q}=const$. In case H depends solely on absolute values of $\vec{p}$ and $\vec{q}$, i. e. $H(\vec{p},\vec{q},t)=H(p,q,t)$, the only conservative value is the moment of momentum $\vec{M}=\vec{q}\times\vec{p}$, which can be checked by differentiation of Eq. (4).

The conservation laws listed here (conservation of energy, of momentum, and coordinate) are characteristic by their tight relation to properties of the system with respect to time and the momentum and coordinate spaces. The energy conservation law is associated with stationarity of the system. The momentum conservation law reflects invariance of its Hamiltonian with respect to arbitrary translations in the coordinate space, while the coordinate conservation law suggests the Hamiltonian translational invariance in the momentum space.

If the Hamiltonian function of a particle is independent of time and the spatial coordinates then the particle is called free. Hamilton's function of a set of free particles is a sum of their individual Hamiltonians, i. e.

$$H(\vec{p}_1,\dots \vec{p}_i,\dots \vec{p}_N)=\sum_{i=1}^{N} H_i(\vec{p}_i) \qquad (12)$$

The specific form of a free particle's Hamiltonian can be found from the demand that H and $\vec{p}$ were components of a Lorentz-invariant four-vector. The dimensionality of Hamilton's function in the CGSE system is that of energy, hence in order that H and $\vec{p}$ could form a four-vector $i\vec{p}$ should be multiplied by a factor of velocity dimensionality. We will denote it c. Thus, a free particle is characterized by the four-vector $(H,\ ic\vec{p})$. Its square should be a positive value that will be denoted as m^2c^4,

$$H^2-c^2p^2=m^2c^4, \qquad (13)$$

where m is called mass.
Eq. (13) yields

$$H=(m^2c^4+c^2p^2)^{1/2}. \qquad (14)$$

Besides, it allows the Hamilton-Jacobi equation to be brought to the form

$$\frac{1}{c^2}\cdot\left(\frac{\partial S}{\partial t}\right)^2-(\nabla S)^2=m^2c^2. \qquad (15)$$

Before advancing further, let us dwell on some purely mathematical problem, noting the results but omitting the derivations for brevity. Consider the equation

$$\hat{L}\left\{\vec{r},t,i\hbar\frac{\partial}{\partial\vec{r}},i\hbar\frac{\partial}{\partial t},\hbar\right\}\psi=0 \qquad (16)$$

where $\hbar$ is a parameter. Let $\hbar$ be small and the operator $\hat{L}$ be a holomorphic function of its last argument (i. e. $\hbar$). By expanding the operator in powers of this argument we can bring Eq. (16) to the form

$$\sum_{k=0}^{\infty}\hbar^k a_k\hat{L}_k\left\{\vec{r},t,i\hbar\frac{\partial}{\partial\vec{r}},i\hbar\frac{\partial}{\partial t},0\right\}\psi=0, \qquad (17)$$

with $\hat{L}_k=\left.\dfrac{\partial^k\hat{L}}{\partial\hbar^k}\right|_{\hbar=0}$.

Eq. (16) cannot be expanded in powers of the $\hbar$ standing in front of the derivatives as the solution is not analytic with respect to this $\hbar$. As is known, the solution can be sought in the form

$$\psi=e^{-\frac{i}{\hbar}S(\vec{r},t)}\left(\sum_{l=0}^{\infty}\hbar^l A_l(\vec{r},t)+\sum_{n=1}^{\infty}\hbar^n\psi^{(n)}(\vec{r},t)\right), \qquad (18)$$

Zeroth-order approximations in $\hbar$ result in the equation for S,

$$L_0\left\{\vec{r},t;\frac{\partial S}{\partial\vec{r}},\frac{\partial S}{\partial t}\right\}=0 \qquad (19)$$

If Eq. (19) is known explicitly, it allows restating the initial equation for ψ in the lowest-order non-vanisting approximation in $\hbar$, i. e.

$$\hat{L}_0\left\{\vec{r},t;i\hbar\frac{\partial}{\partial\vec{r}},i\hbar\frac{\partial}{\partial t}\right\}\psi=0 \qquad (20)$$

Comparison (19) and (20) points to the simple result of $\hat{L}_0$ restoration. Eq. (20) will be named the wave equation and ψ the wave function. The physical object that can be described in terms of a wave function call complete. If the initial form is the Hamilton-Jacobi equation (Eq. (9)), then the corresponding wave equation is

$$i\hbar\frac{\partial\psi}{\partial t}-\hat{H}\left\{\vec{r},t,i\hbar\frac{\partial}{\partial\vec{r}}\right\}\psi=0. \qquad (21)$$

In the case of a free particle with the Hamilton-Jacobi equation like Eq. (14), the wave equation (21) takes the familiar form

$$\hat{\alpha}_4\frac{\partial\psi}{\partial t}-c\hat{\alpha}_l\frac{\partial\psi}{\partial x_l}=\frac{mc^2\hat{\alpha}_4}{\hbar}\psi, \qquad (22)$$

where l=1,2, and 3; summation is meant over the repeated indices; and $\hat{\alpha}_l$, $\hat{\alpha}_4$ are Dirac's matrices. The function ψ in this equation is a bispinor.

The wave equation corresponding to the initial Hamilton-Jacobi equation Eq. (15) is the Klein-Gordon equation,

$$\Delta\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=\frac{m^2c^2}{\hbar^2}\psi. \qquad (23)$$

As founds in experiment, there are particles of zero mass, m=0, for which ψ is a four-dimensional Lorentz vector, $\{\vec{A},i\phi\}$. The field described in terms of this four-vector is the electromagnetic field. ϕ and $\vec{A}$ obey d'Alambert's equations

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0,$$
$$\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0. \qquad (24)$$

Note the constant parameter $\hbar$ is not to appear in the d'Alambert equation. ϕ is known as the scalar electromagnetic potential and $\vec{A}$ the vector potential. Neither the scalar, nor the vector potential have a physical meaning. Measurable values are the electric, $\vec{E}$, and the magnetic, $\vec{H}$, field vectors, related to the potentials as

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\phi,\ \vec{H} = \nabla\times\vec{A} \qquad (25)$$

By applying the $\nabla\times$ (curl) operator to the first of these equations and $\nabla\cdot$ to the second we obtain

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \qquad (26)$$

$$\nabla\cdot\vec{H} = 0 \qquad (27)$$

These equations are known as the first pair of Maxwell equations. They involve vector values that are physically meaningful.

Let us derive the second pair of Maxwell equations. As is known (and is apparent from Eq. (25)), the vector $\left\{i\phi, \frac{1}{c}\vec{A}\right\}$ is not defined uniquely. Equal values of $\vec{E}$ and $\vec{H}$ can be obtained from different $\vec{A}$ and ϕ. Profiting by this, let us impose additional conditions upon $\vec{A}$ and ϕ, e. g.

$$\phi = 0 \text{ and } \nabla\cdot\vec{A} = 0, \qquad (28)$$

then, combining the second equation (24), Eqs. (25) and the equality $\nabla\times\nabla\times\vec{A} = -\Delta\vec{A}$, we arrive at

$$\nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad (29)$$

$$\nabla\cdot\vec{E} = 0. \qquad (30)$$

Finally, let us write the equations determining the interaction of an electromagnetic field with a particle of charge e. The set consists of equations for ϕ and $\vec{A}$ and Dirac equation, viz.

$$\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\vec{j},$$
$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\rho, \qquad (31)$$

$$\left[\gamma_4\left(-i\hbar\frac{\partial}{\partial t} + e\phi\right) + \right.$$
$$\left. + \gamma_l \cdot c\left(\hbar\frac{\partial}{\partial x_l} - i\frac{e}{c}A_l\right) + mc^2\right]\psi = 0. \qquad (32)$$

Here $\vec{j}$ is current of charges

$$j_l = ie\bar{\psi}\gamma_l\psi, \qquad (33)$$

ρ - the charge density

$$\rho = e\bar{\psi}\gamma_4\psi. \qquad (34)$$

In principle, the equations of the last section allow constructing, without additional experiments, the classical and the quantum mechanics and electrodynamics, as well as statistical mechanics and kinetic theory. The same theory includes gravitation fields and elementary particles in a natural way.

Now we can enumerate the basic experimental facts that can be regarded as axioms for theoretical physics.

1. The possibility of Hamiltonian description of physical systems and the energy conservation law (Equation (21).
2. The fact that energy and momentum make up an invariant four-vector, providing the characteristic particle velocities and mass m. These values should be measurable in experiment (Eq. (13)).
3. The possibility of classical-to-quantum mechanics transition, characterized by the parameter $\hbar$ (Plank's constant) that should be measurable in experiment.
4. Ascertainment of the electromagnetic field as a physical reality and derivation of Maxwell's equations (Eqs. (24),(29) and (30)).
5. Establishment of a relation between the electromagnetic field and charged particles. The reality and measurement of the charge e.

Total: 5 axioms.

References

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Об аксиоматическом построении физики

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Делается попытка аксиоматического построения физики. В качестве аксиом принят ряд известных фактов, установленных опытным путем. Постулируется гамильтоновость рассматриваемых систем, без привлечения дополнительных предположений выводятся уравнения классической и квантовой механики и электродинамики.

Про аксіоматичну побудову фізики

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Зроблено спробу аксіоматичної побудови фізики. За аксіоми прийнято ряд відомих фактів, які були встановлені експериментальним шляхом. Постулюється гамільтоновість систем, що розглядаються, без залучення додаткових припущень виводяться рівняння класичної і квантової механіки та електродинаміки.