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Empirical Estimate for the Element Number of a Nonredundant Configuration on a Hexagonal Telescope Aperture

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The empirical upper estimate is obtained for the number of elements of a nonredundant configuration on a given size hexagonal telescope aperture. Nonredundancy means that all the differences between vector radii of the configuration elements ("vector differences") are distinct. The suggested estimate agrees well with the data available and can serve as a guiding line in evaluation of the maximum element number of a large-order nonredundant configuration.

Introduction

The problem of constructing a large-order nonredundant configuration (NRC) on the 2-D telescope aperture is highly pressing for astronomical imaging applications such as radio interferometry and removal of the effects of atmospheric turbulence from ground-based observations at visible and infrared wavelengths [1]. Given the aperture size, it reduces to obtaining the maximum-element NRCs. This issue has been first studied in [2] for the NRCs on square and hexagonal integer grids of small size. NRCs on square grids (also called Golomb squares [3]) of a larger size were then studied in a number of papers, the latest data being given in [4].

A similar problem for hexagonal apertures that is even of more interest for astronomical applications has been as yet studied to a considerably less extent. Two methods for building large-order NRCs on hexagonal grids, both employing planar cyclic difference sets (CDSs), have been proposed. One of these was suggested in [1, 5] as a particular case of constructing uniformly redundant arrays. In a different way, the NRCs on hexagonal grids were obtained in [6, 7] by folding segments with planar CDSs placed on them onto squares and then turning into hexagons.

Building NRCs having the maximum number of elements encounters serious difficulties with larger grids. The NRCs on rectilinear hexagons of radii up to 13 whose element numbers reached or at least were close to maximum were obtained in [6], and also with the method of random search in [8].

To be able to predict maximum element number of the NRC on a grid of large size, its reasonable upper estimate will be required. Estimates for a square grid, though overstated, were obtained in [3, 9], while, to our best knowledge, no one studied this issue for hexagons.

In this paper an attempt is made to fill a want, and an empirical estimate for the NRC element number on a hexagonal grid whose quality is approved by comparing with the available data is suggested.

Estimating the Element Number of a Nonredundant Configuration on a Hexagonal Grid

Take an $n\times n$ square (with the n odd) and place an NRC on it. As shown in [10], by using the transformation with the matrix

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 0 & 1 \end{pmatrix} \qquad (1)$$

the interior domain of this square

$$|x - y| \le (n-1)/2 \qquad (2)$$

can be converted into a rectilinear hexagon of radius $r = (n-1)/2$ (see Fig. 1; the radius means distance between the hexagon centre and its vertex). At this point, because of linearity of transform (1), the NRC arranged in domain (2) of the square turns into the one on this hexagon. Thus, this domain will be dealt instead of the relevant hexagon.

Now, scan the square and obtain a set of dots on a segment of length $N = n^2$. In the general case, the elements of the set obtained on the scan would be located throughout its length, and, at this point, some differences between them would double. However, in our case, when the NRC elements are concentrated in domain (2) of the square, the set on the scan possesses some peculiarities.

To make things clearer, assign numbers to the elements on the square accordingly to those of the relevant elements on the scan. Then one may call a vector difference between two elements on the square as "forward-directed" (type I) or "rearward-directed" (type II) according to the direction of the arrow connecting the "smaller" and the "larger" elements, with respect to the direction of axis x (see Fig. 2, a). It can be seen that owing to the shape of domain (2) representing a stripe stretched ahead and downwards, the vector differences of type I between the NRC elements arranged on it occur much more frequently than those of type II. So, in Fig. 2, a, among the vector differences between the NRC elements, 44 belong to type I and 16 to type II (there are also 6 vertically directed ones).

Further, a pair of distinct vector differences of the same type cannot give equal differences on the scan. Therefore, the number of the two-fold differences on the scan is no more than that of the rearward-directed ones. In fact, it is much smaller (in our example, there are 8 pairs of equal differences $(4, 7, 13, 16, 22, 23, 32, 53)$ on the scan: $4 = 49 - 45 = 71 - 67$, $7 = 45 - 38 = 52 - 45$, etc (see Fig. 2, b)).

Similarly, when scanning the square in Fig. 1, one obtains that among the differences between the 10-element set on the scan there would be 5 two-fold ones.

So, the share of the two-fold differences between the set elements on the scan is small, i. e. a set on the scan thus obtained may possess only a weak redundancy.

Then, when scanning an $n \times n$ square, the NRC elements located in domain (2) cannot fall into some regions of the scan. One might say that these regions (whose total length

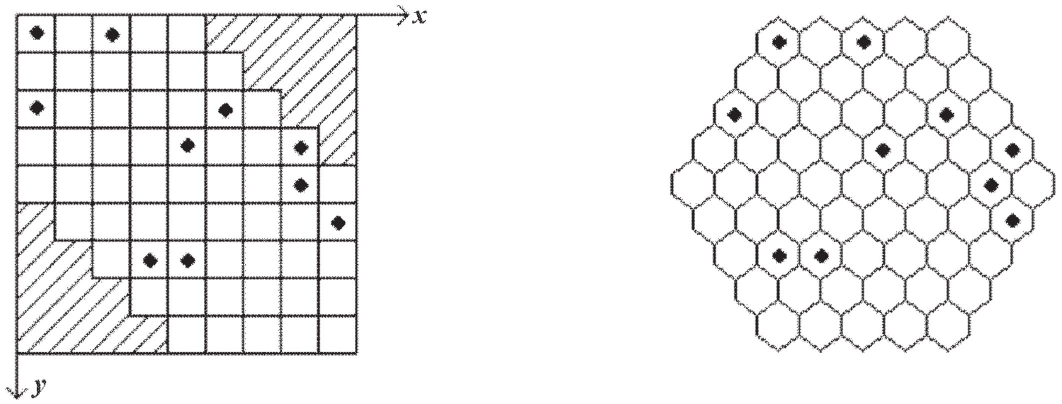

Fig. 1. *An $n \times n$ square and the hexagon of radius $r = (n-1)/2$ obtained from the undashed domain of this square. The NRC on this domain turns into the one on the hexagon*

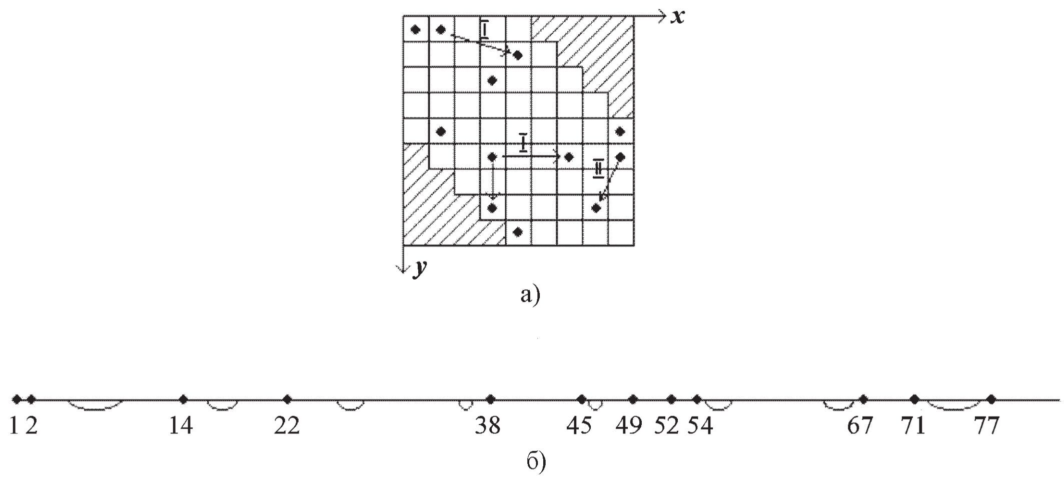

Fig. 2. *(a) A 12-element NRC located in the undashed domain of a 9×9 square. The forward- and rearward-directed vector differences between its elements are denoted by I and II, respectively. The difference between the elements located in the same grid column is denoted by a vertical arrow.*
(b) The set obtained from this NRC by scanning (the figure is scaled down). The regions on the scan forbidden for the set elements are outlined with arcs

equals nearly a quarter of the scan length) are “forbidden” for arranging elements of our set (see Fig. 2, b).

It will be noted that the mentioned peculiarities of such a set on the scan would differ little from the nonredundant set on a segment of equal length; at this point, the elements would differ in number otherwise: the presence of the forbidden regions reduces the number of elements possible, whereas the redundancy increases it. Such reasonings prompt an idea of trying to evaluate the maximum element number of our set by making use of the estimate for the element number (k) of a nonredundant set on a segment of length N [11]:

$$k < \sqrt{N} + \sqrt[4]{N} + 0.5. \qquad (3)$$

There is one-to-one correspondence between the obtained set on the scan and the NRC on the hexagon, therefore, accounting for the relationships $N = n^2 = (2r+1)^2$, inequality (3) can be rewritten in the form

$$k < 2r + \sqrt{2r+1} + 1.5, \qquad (4)$$

or, equivalently, as $k \le k_e$, where k_e is the integer part of the right-hand side of (4).

To check inequality (4) as an upper estimate for the NRC element number on a hexagon of radius r, compare the values of k_e with the data on k given in [6, 8] – see Table.

As can be seen, the estimate fits our data quite well, and we may suppose it also holding for grids of larger radii.

Note that the data on k for grids of radii $r > 13$ also obtained in [6] are essentially less than could be expected from (4). However, the

Table. *Estimate of the NRC element number on hexagonal grids*

r	k	k_e	r	k	k_e
1	4	5	8	21	21
2	7	7	9	22	23
3	9	9	10	25	26
4	12	12	11	27	28
5	14	14	12	28	30
6	16	17	13	30	32
7	19	19			

search made for such grids was far from being exhaustive, and NRCs with larger numbers of elements might exist there.

Conclusion

Estimate (4) for the maximum number of elements of nonredundant configurations on hexagonal grids is suggested. Although obtained non-rigorously, it agrees well with the best values found of their number on hexagons of radii for which the search was adequately made, and provides a guiding line in evaluation of the maximum element number in nonredundant configurations on the grids of larger sizes.

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Эмпирическая оценка числа элементов безызбыточной конфигурации на гексагональной апертуре телескопа

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Получена эмпирическая верхняя оценка числа элементов безызбыточной конфигурации на гесагональной апертуре телескопа заданного размера. Безызбыточность означает, что разности радиус-векторов элементов конфигурации ("векторные разности") все различны. Полученная оценка хорошо согласуется с имеющимися данными и может служить ориентиром при оценке максимального числа элементов безызбыточной конфигурации на решетке больших размеров.

Емпірична оцінка кількості елементів безнадлишкової конфігурації на гексагональній апертурі телескопу

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Отримано емпіричну верхню оцінку кількості елементів безнадлишкової конфігурації на гексагональній апертурі телескопу заданого розміру. Безнадлишковість означає, що різниці радіус-векторів елементів конфігурації ("векторні різниці") усі різні. Отримана оцінка добре погоджується з наявними даними і може слугувати орієнтиром у оцінці максимальної кількості елементів безнадлишкової конфігурації на решітці великих розмірів.