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EVOLUTION EQUATIONS FOR THE TIME-DOMAIN MODES IN LOSSY WAVEGUIDES

O.A. Tretyakov

Kharkov National University, Ukraine;
Gebze Institute of Technology, Turkey,
E-mail: tretyakov@penta.gyte.edu.tr

Complete set of the time-domain modes is presented for a waveguide regular geometrically along its axis Oz. The waveguide under study may have an arbitrary closed singly connected contour L of its cross section. Waveguide surface has the perfect electric conductivity, its volume is filled with a lossy medium with constant electromagnetic parameters ε, μ, σ. Electromagnetic fields of the time-domain modes are products of some functions of the transverse waveguide coordinates, which originate the modal basis, and the modal amplitudes, which are some functions of axial coordinate z, and time t. Modal basis is specified in a general form. Evolution equations for the modal amplitudes are obtained and rearranged to the Klein-Gordon equation, which can be solved easily in compliance with the causality principle.

Introduction

In the classical waveguide theory, existence of a complete set of the *time-harmonic* modes has been established long ago. From that time and hitherto, the waveguide mode concept has become generally accepted in the Frequency Domain (*FD*). Recently, the modal concept was extended over the Time Domain (*TD*) within the frames of the *Evolutionary Approach to Electromagnetics* (*EAE*) [1,2]. While one operates with the classical waveguide modes, study of the time-domain waveforms and signals gives rise usually to essential difficulties. Presence of the ohmic losses in waveguides aggravates the problems. The *EAE* permits either remove these difficulties completely or facilitate them essentially.

In this paper, we study a waveguide regular geometrically along its axis Oz. Contour L of the waveguide cross section is closed and singly connected but it may have rather arbitrary form. The surface is perfectly conducting; the waveguide volume is filled with a lossy medium specified by the linear constitutive relations as

$$\mathcal{D} = \varepsilon_0 \varepsilon \mathcal{E}, \mathcal{B} = \mu_0 \mu \mathcal{H}, \mathcal{J} = \sigma \mathcal{E}, \qquad (1)$$

where ε_0, μ_0 – the free space constants, $\mathcal{D}$, $\mathcal{B}$ – electric and magnetic flux densities, $\mathcal{E} \equiv \mathcal{E}(\mathbf{R}, t)$, $\mathcal{H} \equiv \mathcal{H}(\mathbf{R}, t)$ – electric and magnetic strength vectors, $\mathbf{R}$ – position vector, t – time; real constants ε, μ, and σ are permittivity, permeability, and ohmic conductivity, respectively.

Formulation of the Problem

The system of differential Maxwell's equations

$$\begin{cases} \nabla \times \mathcal{H} = \partial_t \mathcal{D} + \mathcal{J}, & \nabla \times \mathcal{E} = \partial_t \mathcal{B}, \\ \nabla \cdot \mathcal{D} = \rho & \nabla \cdot \mathcal{B} = 0, \end{cases} \qquad (2)$$

should be solved simultaneously with the algebraic boundary conditions, which hold over the contour L as

$$(\mathbf{n} \cdot \mathcal{H})|_L = 0, (\mathbf{l} \cdot \mathcal{E})|_L = 0, (\mathbf{z} \cdot \mathcal{E})|_L = 0, \qquad (3)$$

where $(\mathbf{l}, \mathbf{n}, \mathbf{z})$ – the right-hand triple of the unit vectors: among them, $\mathbf{l}$ is tangential to the contour L the unit vector, $\mathbf{n}$ – the outward unit normal to L, $\mathbf{z}$ – the unit vector oriented along the waveguide Oz-axis.

The solution sought for should belong to the class of quadratically integrable complex valued vector functions specified by the following condition:

$$\int_{t_1}^{t_2} dt \int_{z_1}^{z_2} dz \int_S ds \left(\varepsilon_0 \mathcal{E} \cdot \mathcal{E}^* + \mu_0 \mathcal{H} \cdot \mathcal{H}^* \right) < \infty, \qquad (4)$$

where $(^*)$ means complex conjugation, $0 \le t_1 < t_2 < \infty$, $z_1 < z_2 < \infty$.

Appropriate *initial conditions* for the electromagnetic field sought should be added. The problem must be solved in compliance with the *causality principle*.

It is convenient to split the 3-component position vector $\mathbf{R}$ and the nabla operator ∇ as well onto their projections on the waveguide cross-section S and on Oz-axis as follows

$$\mathbf{R} = \mathbf{r} + \mathbf{z}z, \quad \nabla = \nabla_\perp + \mathbf{z}\partial_z, \qquad (5)$$

where $\mathbf{r}$ is 2-component position vector in the domain S; operator $\nabla_\perp$ acts on the transverse coordinates $(\mathbf{r})$ only. Let's do the same with the 3-component vectors $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{J}$ what yields

$$\mathcal{E} = \mathbf{E} + \mathbf{z}E_z, \mathcal{H} = \mathbf{H} + \mathbf{z}H_z, \mathcal{J} = \mathbf{J} + \mathbf{z}J_z, \qquad (6)$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$ are 2-component projections on S of appropriate three-dimensional vectors. The space-time argument $(\mathbf{R}, t)$ of all the electromagnetic quantities is equivalent now to $(\mathbf{r}, z, t)$.

The *curl* equations from Eqs. (2) should be also projected onto the waveguide cross-section S and Oz-axis. The results of these manipulations can be collected with the *div*-equations as two simultaneous subsystems, namely:

$$\begin{cases} [\nabla_\perp \times \mathbf{z}]H_z = \varepsilon_0\partial_t(\varepsilon\mathbf{E}) + \partial_z[\mathbf{H}\times\mathbf{z}] + \mathbf{J}, \\ \mu_0\partial_t(\mu H_z) = \nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}], \\ \partial_z(\mu H_z) = -\mu\nabla_\perp\cdot\mathbf{H}; \end{cases} \qquad (7)$$

$$\begin{cases} [\mathbf{z}\times\nabla_\perp]E_z = \mu_0\partial_t(\mu\mathbf{H}) + \partial_z[\mathbf{z}\times\mathbf{E}], \\ \varepsilon_0\partial_t(\varepsilon E_z) = \nabla_\perp\cdot[\mathbf{H}\times\mathbf{z}] - J_z, \\ \partial_z(\varepsilon E_z) = -\varepsilon\nabla_\perp\cdot\mathbf{E} + \varepsilon_0^{-1}\rho. \end{cases} \qquad (8)$$

Complete Set of the Time Domain Modes

Eqs. (7) and (8) have two sorts of solutions. One set corresponds physically to the *TE–* time-domain waveguide modes as

$$\begin{aligned} &\mathcal{E}_{\pm m} = \pm\tfrac{1}{\sqrt{\varepsilon_0}}V_{\pm m}^{TE}(z,t)[\nabla_\perp\psi_m\times\mathbf{z}], \\ &\mathcal{H}_{\pm m} = \tfrac{1}{\sqrt{\mu_0}}\big[I_{\pm m}^{TE}(z,t)\nabla_\perp\psi_m + \\ &\qquad + \mathbf{z}h_{\pm m}(z,t)\nu_m^2\psi_m\big], \\ &\mathcal{E}_0 = \mathbf{0}, \mathcal{H}_0 = \mathbf{z}\tfrac{1}{\sqrt{\mu_0}}h_0(z,t); m = 1,2,\ldots. \end{aligned} \qquad (9)$$

Another one represents *TM–* time-domain modes as

$$\begin{aligned} &\mathcal{E}_{\pm n} = \tfrac{1}{\sqrt{\varepsilon_0}}\big[V_{\pm n}^{TM}(z,t)\nabla_\perp\phi_n + \\ &\qquad + \mathbf{z}e_{\pm n}(z,t)\kappa_n^2\phi_n\big], \\ &\mathcal{H}_{\pm n} = \pm\tfrac{1}{\sqrt{\mu_0}}I_{\pm n}^{TM}(z,t)[\mathbf{z}\times\nabla_\perp\phi_n], n = 1,2,\ldots \end{aligned} \qquad (10)$$

In Eqs. (9), potentials $\psi_m \equiv \psi_m(\mathbf{r})$ are solutions of well studied scalar *Neumann* boundary eigenvalue problem for Laplacian, i.e.,

$$\left(\nabla_\perp^2 + \nu_m^2\right)\psi_m(\mathbf{r}) = 0, \quad \tfrac{\partial}{\partial\mathbf{n}}\psi_m(\mathbf{r})\big|_L = 0, \qquad (11)$$

where ν_m are nonnegative eigenvalues, and $\partial/\partial\mathbf{n}$ means the normal derivative over the contour L. In turn, potentials $\phi_n \equiv \phi_n(\mathbf{r})$ are specified by the scalar *Dirichlet* boundary eigenvalue problem for Laplacian as

$$\left(\nabla_\perp^2 + \kappa_n^2\right)\phi_n(\mathbf{r}) = 0, \quad \phi_n(\mathbf{r})|_L = 0, \qquad (12)$$

where positive κ_n originate another set of the eigenvalues. All the solutions $\psi_m(\mathbf{r})$ and $\phi_n(\mathbf{r})$ can be obtained from the problems (11) and (12) accurate within some constant factors. The latters can be specified with making use of some normalization conditions. It is convenient to specify them as

$$\begin{aligned} &(1/S)\int_S\left(\nabla_\perp\psi_m\cdot\nabla_\perp\psi_{m'}^*\right)ds \equiv \\ &\quad\left(\nu_m^2/S\right)\int_S\psi_m\psi_{m'}^*ds = \delta_{mm'}, \\ &(1/S)\int_S\left(\nabla_\perp\phi_n\cdot\nabla_\perp\phi_{n'}^*\right)ds \equiv \\ &\quad\left(\kappa_n^2/S\right)\int_S\phi_n\phi_{n'}^*ds = \delta_{nn'}, \end{aligned} \qquad (13)$$

where $\delta_{mm'}$, $\delta_{nn'}$ are the Kronecker's delta.

Thus, one can consider all the functions $\psi_m(\mathbf{r})$ and $\phi_n(\mathbf{r})$, where $\mathbf{r}\in S$, as already known in the products placed at the right-hand sides of Eqs. (9) and (10). The sets of potentials $\{\psi_m(\mathbf{r})\}$ and $\{\phi_n(\mathbf{r})\}$ are complete in a Hilbert space. It was proved in [1,2], that the sets of functions with argument $(\mathbf{r})$ in Eqs. (9) and (10) originate jointly a waveguide *modal basis*. The scalar functions with argument (z,t) herein are unknown yet. Physically, they are the modal amplitudes of appropriate field components. To obtain a problem for them, one should project Maxwell's equations (in their form of Eqs. (7) and (8) onto the same modal basis. Eqs. (7) and (8) include partial derivatives ∂_t and ∂_z at their right-hand sides. Hence, equations for the modal amplitudes should be partial differential equations with ∂_t and ∂_z. Mathematicians call differential equations with ∂_t as evolution equations.

Evolution Equations for the Modal Amplitudes

TE– Time-Domain Modes

For brevity sake, let's restrict ourselves with studying waveforms, which transfer energy along Oz-axis. In Eqs. (9), one should take upper sign in the doublet $(\pm)$. Longitudinal component of magnetic field $h_{\pm m}(z,t)$ must have upper sign $(+)$ at the subscript as well and be solution of the following equation:

$$\left(\partial_t^2 + 2\gamma\partial_t - c^2\partial_z^2 + c^2\nu_m^2\right)h_{+m}(z,t) = 0, \qquad (14)$$

where $c = 1/\sqrt{\varepsilon_0\mu_0\varepsilon\mu}$, $2\gamma = \sigma/\varepsilon_0\varepsilon$, $m = 1,2,\ldots$. Coefficients $\nu_m > 0$ herein are positive eigenvalues taken from Neumann problem (11). Each one specifies a concrete *TE*–mode. Naturally, it depends on the boundary condition in Eqs. (11): $\nu_m \equiv \nu_m(L)$. So, a form of the waveguide cross section L is present in Eq. (14) implicitly: via numerical the coefficient ν_m^2. Evolution equations for the modal amplitudes of the transverse field components are obtained as direct formulas:

$$I_{+m}^{TE} = \partial_z h_{\pm m}, \quad V_{+m}^{TE} = -(\eta/c)\partial_t h_{+m}, \qquad (15)$$

where $\eta = \sqrt{\mu/\varepsilon}$.

The set of *TE*–modes (9) include a specific one with its amplitude $h_0(z,t)$. While the contour L is singly-connected, it satisfies the following pair of evolution equation: $\partial_t h_0(z,t) = 0$, $\partial_z h_0(z,t) = 0$. Amplitude $h_0(z,t)$ is obviously a constant in the case.

It is convenient to present amplitude $h_{+m}(z,t)$ as

$$h_{+m}(z,t) = e^{-\gamma t} H_{+m}^z(z,t). \qquad (16)$$

Then new unknown function $H_{+m}^z(z,t)$ satisfies well studied Klein-Gordon equation (*KGE*) as

$$\left(\partial_t^2 - c^2\partial_z^2 + \breve{\omega}_m^2\right)H_{+m}^z(z,t) = 0, \qquad (17)$$

where $\breve{\omega}_m = \sqrt{c^2\nu_m^2 - \gamma^2}$ is the cut-off frequency for the time-harmonic waves, which propagate along Oz-axis in the waveguide loaded with the lossy medium.

TM– Time-Domain Modes

TM–like waveforms, which transfer energy *along* Oz-axis, have modal amplitudes of the single transverse component of *magnetic* field in Eqs. (10) as the solutions of equation

$$\left(\partial_t^2 + 2\gamma\partial_t - c^2\partial_z^2 + c^2\kappa_n^2\right)I_{+n}^{TM}(z,t) = 0, \qquad (18)$$

where $c = 1/\sqrt{\varepsilon_0\mu_0\varepsilon\mu}$, $2\gamma = \sigma/\varepsilon_0\varepsilon$, $n = 1,2,\ldots$; eigenvalue $\kappa_n \equiv \kappa_n(L)$ is taken from Dirichlet problem (12). Modal amplitudes of the longitudinal and transverse components of *electric* field are obtained as solutions of simple evolution differential, namely:

$$\begin{aligned} \partial_t e_{+n} + 2\gamma e_{+n} &= -\tfrac{c_0}{\varepsilon} I_{+n}^{TM}(z,t), \\ \partial_t V_{+n}^e + 2\gamma V_{+n}^e &= -\tfrac{c_0}{\varepsilon}\partial_z I_{+n}^{TM}(z,t), \end{aligned} \qquad (19)$$

where $c = 1/\sqrt{\varepsilon_0\mu_0}$ Solutions of Eq. (18) play role of the force terms herein. Similar to (16) substitution as

$$I_{+n}^{TM}(z,t) = e^{-\gamma t} H_{+n}^{\perp}(z,t) \qquad (20)$$

transforms Eq. (18) into *KGE* as well:

$$\left(\partial_t^2 - c^2\partial_z^2 + \hat{\omega}_n^2\right)H_{+n}^{\perp}(z,t) = 0, \qquad (21)$$

where $\hat{\omega}_m = \sqrt{c^2\kappa_m^2 - \gamma^2}$ has physical sense of the cut-off frequencies for the harmonic *TM*–modes of the waveguide loaded by the lossy medium.

KGE has remarkable mathematical properties of symmetry in the sense of the group theory. Physically, they give a wide set of new time-domain waveforms and signals, which are distinct essentially from the classical time-harmonic waves. Some examples will be exhibited at the Workshop.

Main Results

Time-domain waveguide mode problem is considered as the boundary-initial value problem for the system of Maxwell's equations. Modal basis is specified in the general form of the scalar Dirichlet and Neumann boundary eigenvalue problems for transverse part of Laplacian. They give a complete set of functions dependent on the waveguide transverse coordinates. The modal amplitudes are some functions of the axial coordinate z and time t. Problem for them is obtained via projecting Maxwell's equations onto the modal basis. It results in the evolution equations (partial differential equations with ∂_t and ∂_z) like Klein-Gordon equation. In the same way as Maxwell's equations, the latter is invariant with respect to the relativistic Lorentz transformation. It can be solved easily in compliance with the causality principle.

References

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ЭВОЛЮЦИОННЫЕ УРАВНЕНИЯ ДЛЯ ВРЕМЕННЫХ МОД В ВОЛНОВОДАХ С ПОТЕРЯМИ

О.А. Третьяков

Представлен полный набор временных мод для геометрически регулярного вдоль оси Oz закрытого волновода с произвольным односвязным контуром поперечного сечения L. Поверхность волновода – идеально проводящая, объем заполнен проводящей средой с постоянными электромагнитными параметрами ε, μ, σ. Электромагнитные поля временных мод являются произведением некоторых функций поперечных координат, которые образуют модовый базис, и модовых амплитуд, которые являются функциями продольной координаты z и времени t. Модовый базис определен в общем виде. Эволюционные уравнения для модовых амплитуд приведены к виду уравнений Клейна-Гордона, которые легко могут быть решены в соответствии с принципом причинности.

ЕВОЛЮЦІЙНІ РІВНЯННЯ ДЛЯ ЧАСОВИХ МОД У ХВИЛЕВОДАХ ІЗ ВТРАТАМИ

О.О. Третьяков

Представлено повний набір часових мод для геометрично регулярного вздовж осі Oz хвилеводу з довільним однозв'язним контуром поперечного перетину L. Поверхня хвилеводу є ідеально провідною, об'єм заповнений провідним середовищем зі сталими електромагнітними параметрами ε, μ, σ. Електромагнітні поля часових мод є добутком деяких функцій поперечних координат, які утворюють модовий базис, та модових амплітуд, які є функціями подовжньої координати z та часу t. Модовий базис визначений у загальному вигляді. Еволюційні рівняння для модових амплітуд приведені до вигляду рівнянь Клейна-Гордона, які легко можуть бути розв'язані у відповідності до принципу причинності.