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ADELIC QUANTUM COSMOLOGY

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Adelic quantum cosmology is an application of adelic quantum theory to the universe as a whole. Adelic approach takes into account all archimedean and non-archimedean geometries based on real and p-adic numbers, respectively. Calculation of the corresponding adelic wave function of the universe exploits Feynman's path integral method. In this contribution we will give a short review of p-adic numbers and adeles, as well as motivation and formulation of adelic quantum cosmology. Adelic wave functions for a few minisuperspace models will be presented. There is some discreteness of minisuperspaces, which is a consequence of p-adic quantum effects and depends on adelic quantum state of these models.

1. Introduction

According to classical mechanics, space and time are continuous, and distances can be in principle measured by any accuracy. However, in quantum mechanics there is the uncertainty relation $\Delta x \Delta k \geq \hbar/2$ that imposes a restriction on simultaneous measuring of position x and momentum k. Moreover, quantum mechanics combined with general relativity yields [2] $\Delta x \geq l_{pl} = (G\hbar/c^3)^{1/2} \sim 10^{-35} m$, i.e. the Planck length is the minimum one which can be measured. Thus, nothing can be said about the structure of space-time beyond the Planck scale. In fact, this result is derived using concepts of archimedean geometry and real numbers. Due to this reason it seems to be quite natural at the Planck scale to take into account also non-archimedean geometry based on p-adic numbers. Mathematically (Ostrowski theorem), any nontrivial norm on the field of rational numbers Q is equivalent either to the absolute value $|\cdot|_\infty$ or to the p-adic norm $|\cdot|_p$ (p is a prime number). Completions of Q with respect to the absolute value and p-adic norm give the field of real numbers $R \equiv Q_\infty$ and the fields of p-adic numbers Q_p, respectively. Any p-adic number [3, 4] $x \in Q_p$ can be presented as

$$x = p^\gamma \sum_{i=0}^{\infty} x_i p^i, \; x_i = 0, 1, .., p-1, \qquad (1.1)$$

$\gamma \in Z$. If we wish to take into account all possible geometries to study our universe, then a natural mathematical instrument to do that is just adelic theory. An adele [5] a is an infinite sequence

$$a = (a_\infty, a_2, ..., a_p, ...), \qquad (1.2)$$

where $a_\infty \in Q_\infty$ is a real number and $a_p \in Q_p$ is a p-adic number, with restriction $|a_p|_p \leq 1$ for all but a finite number of p. The set of all adeles $\mathcal{A}$ is a ring under componentwise addition and multiplication. An additive character on $\mathcal{A}$ is

$$\chi(xy) = \prod_v \chi_v(x_v y_v) = \exp(-2\pi i x_\infty y_\infty) \times \prod_p \exp(2\pi i \{x_p y_p\}_p), \quad x, y \in \mathcal{A}, \quad (1.3)$$

where $\{a_p\}_p$ denotes the fractional part in expansion (1.1) of a_p. In the case of map $f : Q_p \to C$ (also $Q_\infty \to C$) there is well defined translatory invariant Haar measure dx with the property $d_p(ax) = |a|_p dx$, $a \neq 0$. We use here the Gauss integral

$$\int_{|x|_p \leq p^\gamma} \chi_p(ax^2 + bx) dx = \begin{cases} p^\gamma \Omega(p^\gamma |b|_p), & |4a|_p \leq p^{-2\gamma}, \\ \frac{\lambda_p(a)}{|2a|_p^{1/2}} \chi_p\left(-\frac{b^2}{4a}\right) \times \Omega(p^{-\gamma} |\frac{b}{2a}|_p), & |4a|_p > p^{-2\gamma}, \end{cases} \qquad (1.4)$$

$\gamma \in Z$, where $\lambda_p(a)$ is the arithmetic function for which holds [3] $|\lambda_p(a)|_\infty = 1$, $\lambda(a)\lambda(-a) = 1$, $\lambda(ac^2) = \lambda(a)$, $\lambda_p(a)\lambda_p(b) = \lambda_p(a+b)\lambda_p(a^{-1} + b^{-1})$. The characteristic function Ω in (1.4), defined by $\Omega(u) = 1$ if $u \leq 1$ and $\Omega(u) = 0$, if $u > 1$, plays

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a role of a vacuum state in p-adic quantum mechanics. Since 1987, there has been a significant investigation in construction of physical models with p-adic numbers and adeles (for a review, see [3]).

2. Adelic quantum mechanics

Since field of p-adic numbers Q_p is totally disconnected, there is no possibility to define p-adic "momentum" and "Hamiltonian" operator on appropriate way. This operators in the real case are infinitesimal generators of space and time translations, but in p-adic case these infinitesimal translations become meaningless. In p-adic quantum mechanics [3] also multiplication $\widehat{q}\psi \to x\psi$ (x is a position coordinate, ψ is a wave function) has no meaning because $x \in Q_p$ is a p-adic and $\psi \in C$. But finite transformations are meaningful and the corresponding Weyl and evolution operators are p-adically well defined.

Dynamics of p-adic quantum models is described by a unitary evolution operator $U(t)$ in terms of its kernel $\mathcal{K}$

$$U_p(t)\psi(x'') = \int_{Q_p} \mathcal{K}_t(x'', x')\psi(x')dx'. \quad (2.1)$$

Ordinary and p-adic quantum mechanics can be unified in the form of adelic quantum mechanics which is a triple [6]

$$(L_2(\mathcal{A}), W(z), U(t)), \quad (2.2)$$

where $L_2(\mathcal{A})$ is the Hilbert space of complex valued functions of adelic variables, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathcal{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathcal{A})$.

In adelic approach eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x), \quad (2.3)$$

where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = (E_\infty, E_2, \ldots, E_p, \ldots)$ is the corresponding energy, indicies α and β denote energy levels and their degeneration.

Adelic eigenfunction [6] has the form

$$\psi(x) = \psi_\infty(x_\infty) \prod_{p\in M} \psi_p(x_p) \prod_{p\notin M} \Omega(|x_p|_p), \quad (2.4)$$

where $x \in \mathcal{A}$, $\psi_\infty \in L_2(R)$, $\psi_p \in L_2(Q_p)$ and M is a finite set of primes p.

Kernel of the evolution operator (2.1) is given by the p-adic Feynman path integral

$$\mathcal{K}_p(x'', t''; x', t') = \int_{(x',t')}^{(x'',t'')} \chi_p(-S[q])\mathcal{D}q = \int \chi_p\left(-\int_{t'}^{t''} L(\dot{q}, q)dt\right)\mathcal{D}q(t) \quad (2.5)$$

(with the Planck constant $h = 1$). For the systems with quadratic actions (in the case of n uncoupled degrees of freedom) this p-adic path integral has the form [7]

$$\mathcal{K}_p(x''_\alpha, t''; x'_\alpha, t') = \lambda_p\left(-\frac{1}{2}\frac{\partial^2 \overline{S}}{\partial x'_\alpha \partial x''_\alpha}\right) \times \left|\frac{\partial^2 \overline{S}}{\partial x'_\alpha \partial x''_\alpha}\right|_p^{1/2} \chi_p(-\overline{S}(x''_\alpha, t''; x'_\alpha, t')), \quad (2.6)$$

where $\alpha = 1, \ldots, n$. Note that expression has the same form as in standard quantum mechanics.

3. Adelic quantum cosmology

Adelic quantum cosmology is an application of adelic quantum theory to the universe as a whole. Adelic quantum theory unifies both p-adic and standard quantum theory [6]. In the path integral approach to standard quantum cosmology the starting point is Feynman's idea that the amplitude to go from one state with intrinsic metric h'_{ij}, and matter configuration ϕ' on an initial hypersurface Σ', to another state with metric h''_{ij}, and matter configuration ϕ'' on a final hypersurface Σ'', is given by a functional integral of the form

$$\langle h''_{ij}, \phi'', \Sigma'' | h'_{ij}, \phi', \Sigma' \rangle_\infty = \int \mathcal{D}(g_{\mu\nu})_\infty \mathcal{D}(\Phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]), \quad (3.1)$$

over all four-geometries $g_{\mu\nu}$, and matter configurations Φ, which interpolate between the initial and final configurations [8]. The $S_\infty[g_{\mu\nu}, \Phi]$ is the usual Einstein-Hilbert action

$$S[g_{\mu\nu}, \Phi] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K - \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)] \quad (3.2)$$

for the gravitational field and matter fields Φ. To perform p-adic and adelic generalization we first make p-adic counterpart of the action (3.2) using form-invariance under change of real to the p-adic number fields [2]. Then we generalize (3.1) and introduce p-adic complex-valued cosmological amplitude

$$\langle h''_{ij}, \phi'', \Sigma'' | h'_{ij}, \phi', \Sigma' \rangle_p = \int \mathcal{D}(g_{\mu\nu})_p \mathcal{D}(\Phi)_p \chi_p(-S_p[g_{\mu\nu}, \Phi]). \qquad (3.3)$$

The space of all 3-metrics and matter field configurations on a 3-surface is called superspace, which is the configuration space in quantum cosmology. Superspace has infinite dimensions with a finite number of coordinates $(h_{ij}(\vec{x}), \phi(\vec{x}))$ at each point $\vec{x}$ of the 3-surface. In practice, the work with these infinite dimensions is impossible. One useful approximation therefore is to truncate the infinite degrees of freedom to a finite number $q^\alpha(t)$, $(\alpha = 1, 2, ...n)$. In this way one obtains a particular minisuperspace model. Usually, one restricts the 4-metric to be of the form $ds^2 = -N^2(t)dt^2 + h_{ij}dx^i dx^j$, where $N(t)$ is the laps function. For such minisuperspaces, functional integrals (3.1) and (3.3) are reduced to functional integration over 3-metrics, matter configurations and to one usual integral over the laps function. If one takes boundary condition $q^\alpha(t'') = q^{\alpha''}$, $q^\alpha(t') = q^{\alpha'}$ then integrals in (3.1) and (3.3), in the gauge $\dot{N} = 0$, are standard and p-adic minisuperspace propagators, respectively. In this case, for the v- adic minisuperspace propagator (unifies standard and p-adic), we have

$$\langle q^{\alpha''} | q^{\alpha'} \rangle_v = \int dN \mathcal{K}_v(q^{\alpha''}, N | q^{\alpha'}, 0), \qquad (3.4)$$

where

$$\mathcal{K}_v(q^{\alpha''}, N | q^{\alpha'}, 0) = \int \mathcal{D}q^\alpha \chi_v(-S_v[q^\alpha]) \qquad (3.5)$$

is an ordinary quantum-mechanical propagator between fixed q^α and N, and index $v = \infty, 2, 3, \ldots, p, \ldots$ denotes real and p-adic cases.

A necessary condition to construct an adelic model is existence of the p-adic (vacuum) state

$$\prod_{\alpha=1}^{n} \Omega(|q^\alpha|_p),$$

which satisfies equation

$$\prod_{\alpha=1}^{n} \int_{|q^{\alpha'}|_p \le 1} \mathcal{K}_p(q^{\alpha''}, N | q^{\alpha'}, 0) dq^{\alpha'} = \prod_{\alpha=1}^{n} \Omega(|q^{\alpha''}|_p) \qquad (3.6)$$

for all but a finite number of p. The corresponding adelic eigenstates have the form (2.4).

4. Some adelic minisuperspace models

To illustrate the above approach we shall consider some one and two dimensional minisuperspace models.

4.1. Adelic de Sitter model

The de Sitter model is in quantum cosmology the simplest nontrivial exactly soluble model. This model is given by the Einstein-Hilbert action with cosmological term (3.2) without matter fields, and by the Robertson-Walker metric

$$ds^2 = \sigma^2(-N^2(t)dt^2 + a^2(t)d\Omega_3^2) \qquad (4.1)$$

where $\sigma^2 = \frac{2G}{3\pi}$ and $a(t)$ is a scale factor. Instead of (4.1) we prefer metric in the form

$$ds^2 = \sigma^2\left(-\frac{N^2(t)}{q(t)}dt^2 + q(t)d\Omega_3^2\right), \; q(t) > 0, \qquad (4.2)$$

which was considered in the real case [9], and in the p-adic and adelic [10] (in the p-adic generalization of the non-boundary Hartle-Hawking approach) case, because it leads to the quadratic actions. The corresponding adelic action for this one-dimensional minisuperspace model contains

$$S_v[q] = \frac{1}{2}\int_{t'}^{t''} dtN\left(-\frac{\dot{q}^2}{4N^2} - \lambda q + 1\right), \qquad (4.3)$$

where $\lambda = \frac{\Lambda\sigma^2}{3}$. The classical equation of motion (in the gauge $\dot{N} = 0$) $\ddot{q} = 2\lambda$ with the boundary conditions $q(0) = q', q(T) = q''$ $(T = t'' - t')$ has solution

$$q(t) = \lambda t^2 + (\frac{q'' - q'}{T} - \lambda T)t + q', \qquad (4.4)$$

and the corresponding classical action is

$$\overline{S}(q'', T | q', 0) = \frac{\lambda^2 T^3}{24} - [\lambda(q' + q'') - 2]\frac{T}{4} - \frac{(q'' - q')^2}{8T} \qquad (4.5)$$

Since the action is quadratic, then the kernel (3.5) is

$$\mathcal{K}_v(q'', T | q', 0) = \frac{\lambda_v(-8T)}{|4T|_v^{1/2}} \chi_v(-\overline{S}_v). \qquad (4.6)$$

One can show, applying formula (3.6) and (1.4), existence of a necessary p-adic vacuum state in the form Ω function, which is

$$\Psi_p(q,T) = \begin{cases} \Omega(|q|_p), & |T|_p \le 1, p \ne 2, \\ \Omega(|q|_2), & |T|_2 \le \frac{1}{2}, p = 2, \end{cases} \quad (4.7)$$

under condition $\lambda = 4 \cdot 3 \cdot l$, $l \in Z$.

4.2. *Model with cosmological constant in 3 dimensions*

This model in the real case is considered in the paper [11]. Its metric is

$$ds^2 = \sigma^2(-N^2(t)dt^2 + a^2(t)(d\theta^2 + \sin^2\theta d\varphi^2)), \quad (4.8)$$

with $\sigma = G$. Classical equation of motion $\ddot{a} - N^2 a\lambda = 0$ has the solution

$$a(t) = \frac{1}{2\sinh(N\sqrt{\lambda})}\Big((a'' - a'e^{-N\sqrt{\lambda}})e^{N\sqrt{\lambda}t} + (a'e^{N\sqrt{\lambda}} - a'')e^{-N\sqrt{\lambda}t}\Big), \quad (4.9)$$

with the boundary conditions $a(0) = a'$, $a(1) = a''$. For the classical action it gives

$$\bar{S}(a'', N|a', 0) = \frac{1}{2\sqrt{\lambda}}\Big[N\sqrt{\lambda} + \lambda\Big(\frac{2a''a'}{\sinh(N\sqrt{\lambda})} - \frac{a'^2 + a''^2}{\tanh(N\sqrt{\lambda})}\Big)\Big]. \quad (4.10)$$

Quantum-mechanical propagator has the form

$$\mathcal{K}_v(a'', N|a', 0) = \frac{\lambda_v(-2\sinh N)}{|\lambda^{-1/2}\sinh(N\sqrt{\lambda})|_v^{1/2}} \times \chi_v(-\bar{S}(a'', N|a', 0)). \quad (4.11)$$

For this model also exists p-adic vacuum state

$$\Psi_p(a,N) = \begin{cases} \Omega(|a|_p), & |N|_p \le 1, p \ne 2, \\ \Omega(|a|_2), & |N|_2 \le \frac{1}{4}, p = 2, \end{cases} \quad (4.12)$$

with conditions $|\lambda|_p \le 1$ and $|\lambda|_2 \le 2$.

4.3. *Some two dimensional models*

There exists a class of two-dimensional minisuperspace models which after some transformations have the form of two oscillators [12, 13]. These models are: the isotropic Friedmann model with conformally and minimally coupled scalar field and the anisotropic vacuum Kantowski-Sachs model. For all these three models action may be written as

$$S = \frac{1}{2}\int_0^1 dtN\left[-\frac{\dot{x}^2}{N^2} + \frac{\dot{y}^2}{N^2} + x^2 - y^2\right], \quad (4.13)$$

i.e. this is the action for two oscillators, but one of them has a negative energy. This expression leads to the propagator

$$\mathcal{K}_v(y'', x'', N|y', x', 0) = \frac{1}{|N|_v}\chi_v \times \left(\frac{x'^2 + x''^2 - y'^2 - y''^2}{2\tan N} + \frac{y'y'' - x'x''}{\sin N}\right) \quad (4.14)$$

The linear harmonic oscillator is well analyzed system from real as well as from p-adic point of view. One can show that in the p-adic region of convergence of analytic functions $\sin N$ and $\tan N$, which is $G_p = \{N \in Q_p : |N|_p \le |2p|_p\}$, exists vacuum state $\Omega(|x|_p)\ \Omega(|y|_p)$.

5. Conclusion

For some standard minisuperspace models, we constructed the corresponding p-adic and adelic minisuperspace models. For all these models there exist adelic wave functions of the form

$$\Psi(q^1, ..., q^n) = \prod_{\alpha=1}^{n} \Psi_\infty(q_\infty^\alpha) \prod_p \prod_{\alpha=1}^{n} \Omega(|q_p^\alpha|_p), \quad (5.1)$$

where $\Psi_\infty(q_\infty^\alpha)$ are the corresponding wave functions of the universe in standard cosmology. Adopting the usual probability interpretation of the wave function (5.1) in rational points of q^α, we have

$$|\Psi(q^1, ..., q^n)|_\infty^2 = \prod_{\alpha=1}^{n} |\Psi_\infty(q^\alpha)|_\infty^2 \prod_p \prod_{\alpha=1}^{n} \Omega(|q^\alpha|_p),$$

because $(\Omega(|q^\alpha|_p))^2 = \Omega(|q^\alpha|_p)$. As a consequence of Ω-function properties we have

$$|\Psi(q^1, \ldots, q^n)|_\infty^2 = \begin{cases} |\Psi_\infty(q^\alpha)|_\infty^2, & q^\alpha \in Z, \\ 0, & q^\alpha \in Q \backslash Z. \end{cases} \quad (5.2)$$

This result (as those in [1]) leads to the discretization of the minisuperspace coordinates q^α, because probability is nonzero only in the integer points of q^α. Note that this kind of discreteness depends on adelic quantum state of the universe. When system is in some excited state, the sharpness of the discrete structure dissapears and minisuperspace demonstrates usual continues properties.

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АДЕЛЬНАЯ КВАНТОВАЯ КОСМОЛОГИЯ

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Адельная квантовая космология является приложением адельной квантовой теории ко Вселенной как к целому. Адельный подход учитывает все архимедовы и не архимедовы геометрии, основанные на действительных и p - адельных числах, соответственно. Нахождение соответствующей волновой функции Вселенной использует интегральный метод Фейнмана. В этой работе мы даем краткий обзор p -адельных чисел и аделей, а также обоснование и формулировку адельной квантовой космологии. Найдена адельная волновая функция для нескольких минипространственных моделей. Существует дискретность минипространств, являющаяся следствием p -адельных квантовых эффектов и зависящая от адельного квантового состояния этих моделей.

АДЕЛЬНА КВАНТОВА КОСМОЛОГІЯ

В. Драгович, Л. Несік

Адельна квантова космологія є деяким застосуванням адельної квантової теорії до Всесвіту у цілому. Адельний підхід бере до уваги усі архімедові та неархімедові геометрії, що засновані на дійсних та p - адельних числах, відповідно. Для знаходження відповідної хвильової функції використовується інтегральний метод Фейнмана. У цій роботі ми даємо загальний огляд p -адельних чисел та аделів, а також обгрунтування та формулювання адельної квантової космології. Знайдена адельна хвильова функція для кількох мініпросторових моделей. Існує дискретність мініпросторів, яка є наслідком p - адельних квантових ефектів, та залежить від адельного квантового стану цих моделей.