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QUANTUM EFFECTS IN A FLUCTUATING BLACK HOLE GEOMETRY

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We study the propagation of null rays and massless fields in a black hole fluctuating geometry. The metric fluctuations are induced by a small oscillating null incoming flux of energy which is described by a statistical ensemble. The stochastic variables are the phases and the amplitudes of its Fourier modes. By integrating over these variables, we obtain that the field obeys an effective propagation which enjoys the following properties. The amplitude of the metric fluctuations defines a critical length: Smooth wave packets with respect to this length are not significantly affected when they are propagated forward in time. Concomitantly, we find that the asymptotic properties of Hawking radiation are not severely modified. However, backward propagated wave packets are strongly affected by the metric fluctuations once their blue shifted frequency reaches the inverse critical length.

1. Introduction

In his original derivation of black hole radiance, Hawking [1] considered the propagation of a linear quantized field in a classical background geometry, that of a collapsing body. In this framework, one neglects the fluctuations of the geometry. Besides quantum mechanical fluctuations of the gravitational field itself, there exist also metric fluctuations induced by the quantum fluctuations of other fields. The latter can be approximatively described by introducing stochastic noise sources in the right-hand side of the Einstein equations [2-5].In this description, one therefore deals with a stochastic ensemble of fluctuating geometries. Our aim is to study the propagation of a massless field in such an ensemble.

To describe the metric fluctuations near the black hole horizon we use a model similar to that considered by York [6]. It is based on the hypothesis that off-shell fluctuations are driven by a small oscillating flux of energy of an infalling null fluid. In our model, the metric fluctuations are represented by a linear superposition with different frequencies. Stochasticity comes into the picture by assuming that the amplitudes and phases of each mode are stochastic variables. Therefore the expectation value of any observable is obtained averaging over these variables.

In this paper[3], we consider a stochastic ensemble of metric fluctuations. We show that light propagation in a stochastic metric indeed leads to an effective truncated theory near the event horizon. More precisely, we obtain the following. First the critical length ω_c^{-1} is determined by the amplitude of the metric fluctuations. Secondly, as far as forward in time propagation is concerned, the evolution of smooth wave packets (where smooth means that their in-frequency content is much below ω_c) is affected only slightly by the metric fluctuations. Thirdly, backward in time propagation of wave packets representing Hawking quanta is dramatically modified only when the blue shift factor brings their frequency close to ω_c. In this regime, the amplitude of the wave packet is rapidly dissipated (backward in time!).

2. Fluctuating black-hole geometry

2.1. *Metric ansatz*

Let us consider spherical modes of metric fluctuations propagating in a spherically symmetric background. The most general spherical metric can be written in

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the form

$$dS^2 = \gamma_{AB}\, dx^A\, dx^B + \mathcal{R}^2\, d\omega_2^2\,, \qquad (2.1)$$

where $A, B = 0, 1$, and γ_{AB} and $\mathcal{R}$ are functions of x^A. The metric of a black hole of mass M formed by the collapse at $v = 0$ of a massive null shell with mass M is

$$dS^2 = (4M)^2 ds^2\,,$$
$$ds^2 = -A\, dv^2 + 2dv\, dr + r^2 d\omega_2^2\,, \qquad (2.2)$$

where in the absence of fluctuations

$$A = A_0(r, v) = 1 - \frac{\vartheta(v)}{2r}\,, \qquad (2.3)$$

$\vartheta(v)$ being the Heaviside step function equal to 1 for positive argument. For further convenience we have introduced the dimensionless coordinates (v, r), so that $\mathcal{R} = \triangle\mathcal{M}\nabla$ and $4Mv$ are the radius and the advanced time in units where $G = c = 1$.

The most general metric perturbation preserving the form (2.1) of the metric is described by four functions of x^A: δr and $\delta\gamma_{AB}$. The remaining coordinate gauge freedom is generated by infinitesimal coordinate transformations $\xi^A(x)$. We fix the gauge by putting

$$\delta r = 0\,, \qquad \delta\gamma_{rr} = 0\,. \qquad (2.4)$$

The perturbed metric can be written in the form

$$ds^2 = \Psi(-A\, dv^2 + 2dv\, dr) + r^2 d\omega_2^2\,, \qquad (2.5)$$

with

$$\Psi = 1 + \delta\Psi\,, \qquad A = A_0 + \delta A\,. \qquad (2.6)$$

It is evident that the 2-dimensional conformal factor Ψ does not enter the equations for the propagation of radial null rays. In Section 3.1 we shall demonstrate that, for s-modes, Ψ also drops out of the 4-dimensional Dalambertian. Hence only the function A will be relevant for us.

To further simplify the problem, we assume that the metric fluctuation δA is composed only of infalling radial null modes. Thus it is of the form

$$\delta A = -\frac{1}{2r}\,\vartheta(v)\,\mu(v)\,. \qquad (2.7)$$

so that the perturbed metric is given by (2.5) with

$$A = A_0 + \delta A = 1 - \frac{\vartheta(v)[1 + \mu(v)]}{2r}\,. \qquad (2.8)$$

For $\Psi = 1$ this is a Vaidya metric. The function $\mu(v)$ encodes the light-like infalling fluctuations. As in eq. (2.3), the step function in relation (2.8) indicates that the black hole results from the gravitational collapse at $v = 0$ of a massive null shell with mass M, and that there are no fluctuations prior to the collapse of the null shell. Therefore spacetime is flat to the past of the null shell.

2.2. *Stochastic variables*

To introduce the stochastic variables in simple terms, we postulate that $\mu(v)$ possesses a discrete[4] and non-degenerate Fourier decomposition:

$$\mu(v) = \sum_\omega [\mu_1^\omega \sin(\omega v) + \mu_2^\omega \cos(\omega v)]$$
$$= \sum_\omega \mu_0^\omega \sin(\omega v + \phi_\omega)\,. \qquad (2.9)$$

For further simplicity we also assume that the (real) amplitudes μ_1^ω and μ_2^ω are independent stochastic variables characterized by the same distribution. Thus, there is no preferred value of the phase ϕ_ω in the $(\mu_1^\omega, \mu_2^\omega)$ plane. In this case, $\widetilde{\rho}_\omega(\mu_0^\omega)$, the distribution function for the amplitude $\mu_0^\omega = \sqrt{\mu_2^2 + \mu_2^2}$, satisfies the normalization condition

$$\int_0^{2\pi} d\phi_\omega \int_0^\infty d\mu_0^\omega\, \mu_0^\omega\, \widetilde{\rho}_\omega(\mu_0^\omega) = 1\,. \qquad (2.10)$$

Later we shall assume that the distribution $\widetilde{\rho}_\omega(\mu_0^\omega)$ is a Gaussian whose dispersion is equal to $\widetilde{\sigma}_\omega$. Our last assumption concerns the amplitudes of the metric fluctuations: we shall assume that they are much smaller than the gravitational radius of the black hole. In our dimensionless units, this gives $\widetilde{\sigma}_\omega \ll 1$. This should be true for black holes of mass M much greater than the Planck mass m_{Pl} (in [6], the estimate dimensionless amplitude scales as $\widetilde{\sigma} \sim (m_{\text{Pl}}/M)$, whereas in [9] $\widetilde{\sigma} \sim (m_{\text{Pl}}/M)^{4/3}$).

In this paper, an important simplification follows from the fact that we shall deal with observables depending on the fluctuating geometry which obey the following factorization condition

$$Q = \prod_\omega Q_\omega(\mu_0^\omega, \phi_\omega)\,. \qquad (2.11)$$

For these observables, given our hypothesis of stochastic independence, we can consider each sector labeled

[4] A discrete spectrum arises for example in York's approach [6] based on the quasi-normal modes of the black-hole metric. However, the spectrum due to other fields can be continuous. The results of the present paper can be easily adopted to this case. It is sufficient to replace the discrete sum, $\sum_\omega$, by an integral, $\int d\omega\, \nu(\omega)$, where $\nu(\omega)$ is the number density of fluctuation modes.

by ω separately. It is then useful to introduce the successive averages:

$$\overline{Q}_\omega(\mu_0^\omega) = \frac{1}{2\pi}\int_0^{2\pi} d\phi_\omega\ Q_\omega(\mu_0^\omega, \phi_\omega)\,, \quad (2.12)$$

$$<Q_\omega>_\omega = 2\pi \int_0^\infty d\mu_0^\omega\ \mu_0^\omega\ \widetilde{\rho}_\omega(\mu_0^\omega)\ \overline{Q}_\omega(\mu_0^\omega)\,, \quad (2.13)$$

$$\ll Q \gg = \prod_\omega <Q_\omega>_\omega\,. \quad (2.14)$$

The first equality gives $\overline{Q}_\omega$, the average of Q_ω over the stochastic phase ϕ_ω. The second one gives the result of averaging $\overline{Q}_\omega$ over the amplitude μ_0^ω. Finally, the overall ensemble average of the observable Q, is given by (2.14). The order in this averaging procedure follows from the fact that to perform the first average, one simply has to assume that there is no preferred direction in ϕ_ω. For the second instead, we need to choose the distribution $\widetilde{\rho}_\omega$. And for the third one, we must know the whole spectrum.

Because of the factorizability of the operators and their ensemble averages, it will be sufficient to consider only a single fluctuation mode. To simplify the notations we shall drop the index ω in the amplitude and in the phase. That is we shall work in the fluctuating geometry (2.2), (2.8) with

$$\mu(v) = \mu_0 \sin(\omega v + \phi)\,, \quad (2.15)$$

with $\mu_0 \ll 1$. We call the metric (2.2), (2.8) with $\mu(v)$ given by (2.15) a realization of the fluctuating geometry. By averaging over ϕ and μ_0 we thus assume that we are dealing with an ensemble of such realizations.

2.3. *Null ray propagation in a fluctuating geometry*

We first study the propagation of radial null rays in the fluctuating black hole geometry (2.1). In-going rays are given by $v =$ const and out-going rays obey the equation

$$A\,dv = 2\,dr\,. \quad (2.16)$$

In order to solve this equation, we use a method of perturbations and write

$$r = r(v) = R(v) + \rho(v) + \varpi(v) + \ldots\,. \quad (2.17)$$

$R(v)$ is the solution of equation (3.1) in the absence of fluctuations, and $\rho(v)$ and $\varpi((v)$ are respectively the first and second order perturbation in μ_0. Higher order corrections are denoted by dots. In what follows, we shall also often use the dimensionless versions of R, ρ and ϖ which we mark by a tilde.

The equation for out-going rays in the unperturbed metric, $[\ (\ )^{\cdot} \equiv d/dv\ ]$

$$\dot{R} = \frac{1}{2}\left(1 - \frac{\vartheta(v)}{2R}\right) \quad (2.18)$$

can be easily integrated. Let us choose the value of the retarded time u and denote by $r = R(v;u)$ the unperturbed trajectory of a radial ray which arrives to $\mathcal{J}^+$ at the chosen time u. This trajectory for $v > 0$ can be found by solving the equation

$$u = v - 2R_* = \text{const}\,. \quad (2.19)$$

Here

$$2R_* = 2R - 1 + \ln(2R - 1) \quad (2.20)$$

is the dimensionalless tortoise radial coordinate.

The equations for the perturbations $\rho(v)$ and $\varpi(v)$ are obtained by linearizing (2.16). Both functions obey the same equation

$$\dot{f} = \frac{1}{4R^2}f + F\,, \quad (2.21)$$

for $v > 0$ and

$$\dot{f} = 0\,, \quad (2.22)$$

for $v < 0$. For the first order perturbation, one has

$$f = \rho\,, \qquad F = -\frac{1}{4R}\mu\,, \quad (2.23)$$

and for the second order perturbation

$$f = \varpi\,, \qquad F = \frac{1}{4R^2}\mu\rho - \frac{1}{4R^3}\rho^2\,. \quad (2.24)$$

In these equations, the retarded time u is a fixed parameter which specifies the unperturbed ray under consideration and $R = R(v;u)$.

2.4. *Perturbed horizon*

Before discussing the general solution of the equations for the perturbations $\rho(v)$ and $\varpi(v)$ we discuss the particular solution which describes the event horizon in the fluctuating geometry.

First notice that $R = 1/2$ satisfies the unperturbed equation (2.18). This degenerate solution describes an outgoing null ray propagating along the unperturbed event horizon. Starting with this solution we easily obtain the following solutions for the dimensionless perturbations

$$\rho_{\text{EH}} = \frac{\mu_0}{2}\,\frac{\omega\cos(\psi) + \sin(\psi)}{1 + \omega^2}\,, \quad (2.25)$$

$$\varpi_{\rm EH} = \mu_0^2 \times \frac{2\omega^2(2-\omega^2)\cos(2\psi) + \omega(1-5\omega^2)\sin(2\psi)}{2(1+\omega^2)^2(1+4\omega^2)}. \tag{2.26}$$

where $\psi = \omega v + \phi$. It is easy to see that after averaging over the phase ϕ both quantities, ρ and ϖ, vanish. It means that the average position of the fluctuating horizon up to the second order remains unchanged.

It is also interesting to compute the modified value of the surface area $\mathcal{A}$ of the event horizon. When averaging over the phase ϕ, we find

$$\overline{\mathcal{A}} \equiv 4\pi M^2 \overline{(r_{\rm EH}^2)} = 4\pi(\overline{R^2} + \overline{\rho^2}) = 16\pi M^2 \left[1 + \frac{\mu_0^2}{2(1+\omega^2)}\right]. \tag{2.27}$$

Similarly the average value of the surface gravity in the fluctuating geometry is

$$\overline{\kappa} \equiv \frac{1}{4M}\overline{\left(\frac{1+\mu}{r_{\rm EH}^2}\right)} = \frac{1}{4M}\left[1 + 4(3\overline{\rho^2} - \overline{\rho\mu})\right] = \frac{1}{4M}\left[1 + \frac{\mu_0^2}{2(1+\omega^2)}\right]. \tag{2.28}$$

Upon computing the modifications of the Hawking flux, we shall see that this 'renormalized' surface gravity will determined the modified temperature.

2.5. *Late time regime*

In the late time regime, i.e. $u \gg 1$, the relation between the moment v of advanced time when the null ray was emitted from $\mathcal{J}^-$ and the retarded time u when it reaches $\mathcal{J}^+$ in the linear approximation can be written as follows

$$w = W_\phi(u), \tag{2.29}$$

$$W_\phi(u) = w_0 \sin(\phi + \phi_0) + e^{-u}. \tag{2.30}$$

Here we have introduced for later convenience

$$w = -1 - v, \qquad w_0 = \frac{\mu_0}{\sqrt{1+\omega^2}}, \qquad \phi_0 = \arctan \omega. \tag{2.31}$$

For a given realization of the geometry and to first order in μ_0 the event horizon is given by the equation

$$r_{{\rm EH},\phi}(v) = \frac{1}{2}[1 + w_0 \sin(\omega v + \phi + \phi_0)]. \tag{2.32}$$

It crosses the collapsing null shell, $v = 0$ or $w = -1$, at the radius

$$r_{{\rm EH},\phi}(0) = \frac{1}{2}[1 + w_0 \sin(\phi + \phi_0)]. \tag{2.33}$$

Being traced backward in time the null geodesic giving rise to the horizon enters the flat spacetime region inside the collapsing shell, bounces at $r = 0$, and finally reaches $\mathcal{J}^-$ with w lying in the domain

$$w \in (-w_0, w_0). \tag{2.34}$$

Therefore, for all ϕ, radial null rays emitted from $\mathcal{J}^-$ at advanced time $w < -w_0$ fall into the singularity $r = 0$. On the contrary, radial null rays emitted with $w > w_0$ always reach $\mathcal{J}^+$. The time of their arrival to $\mathcal{J}^+$ lies in the interval

$$u \in (u_-, u_+), \tag{2.35}$$

where $u_\pm$ are

$$u_\pm = -\ln(w \mp w_0). \tag{2.36}$$

Finally, the rays emitted in the interval $-w_0 < w < w_0$ reach $\mathcal{J}^+$ only for some values of ϕ. Since for w in this interval there always exists a phase such that the ray propagates along the horizon, the moment of arrival at $\mathcal{J}^+$ varies from $u_- = -\ln(w + w_0)$ to $u_+ = \infty$.

3. Wave propagation in a fluctuating geometry

3.1. *Wave propagation in a given realization of the geometry*

Let us consider propagation of s-modes of a minimally coupled massless scalar field χ. We introduce as usual $\varphi = r\chi$. Then, the four-dimensional Dalambertian equation $\Box\chi = 0$ when computed in the fluctuating metric (2.5) gives, see e.g [10],

$$\left[{}^{(2)}\Box - \frac{\partial_r A}{r}\right]\varphi = 0. \tag{3.1}$$

Hence Ψ defined in eq. (2.5) plays no role. Moreover, upon neglecting the centrifugal quantum potential, one obtains the equation

$${}^{(2)}\Box\varphi = 2\partial_v\partial_r\varphi + \partial_r(A\partial_r\varphi) = 0. \tag{3.2}$$

When adopting this equation, one works in the geometrical optics approximation. Therefore, the spherically symmetric perturbations of the metric affect the global properties of the solutions of (3.2) only through the gluing of the null characteristics encoded in $W_\phi(u)$ given in eq. (2.30).

We shall denote the value of the solutions of (3.2) on $\mathcal{J}^\pm$ by a capital letter

$$\Phi^\pm = \varphi|_{\mathcal{J}^\pm}.$$

Using the coordinate w defined in (2.31) we call $\Phi^-(w)$ the initial value (or image) of the solution φ on $\mathcal{J}^-$, and $\Phi^+(u)$ the final value (or image) of φ on $\mathcal{J}^+$. For a fixed geometry, i.e. for a fixed μ_0 and ϕ, the knowledge of $\Phi^-(w)$ uniquely determines its image on $\mathcal{J}^+$. However, for backward propagation from $\mathcal{J}^+$ to $\mathcal{J}^-$, this is not true since $\mathcal{J}^+$ is not a complete Cauchy surface.

3.2. *Scattering operator in a fluctuating black-hole geometry*

In order to simplify calculations it is convenient to introduce a new coordinate on $\mathcal{J}^+$

$$y = e^{-u} . \tag{3.3}$$

The reason is the following. In terms of y, relation (2.30) is linear equation

$$w = W_\phi(y) \equiv y + w_0 \sin(\phi + \phi_0) . \tag{3.4}$$

Therefore, the effect of the metric fluctuations is simply to shift y with respect to w. The image Φ^+ of a wavepacket on $\mathcal{J}^+$ can be considered as a function of u or of y. In order to avoid confusion, we shall keep the notation $\Phi^+(u)$ for the function of u and shall use the notation $\Phi^+(y)$ whenever the image is considered as a function of y. For any realization of the geometry we have

$$\Phi^+_\phi(y) = \Phi^-(y + w_0 \sin(\phi + \phi_0))\, \Phi^-(y) . \tag{3.5}$$

This relation can be rewritten as

$$\Phi^+_\phi = \mathsf{D}_\phi \Phi^- , \tag{3.6}$$

where

$$\mathsf{D}_\phi = e^{w_0 \sin(\phi+\phi_0)\partial} . \tag{3.7}$$

Notice that this shift operator bears some similarities with that introduced in [11]. Here ∂ is the operator of differentiation with respect to the argument of the function.

Let us first perform the average over the phase. Using the integral representation of the Bessel function of zero index

$$J_0(b) = \frac{1}{\pi} \int_0^\pi d\phi\, e^{ib\cos\phi} , \tag{3.8}$$

we get

$$\overline{\mathsf{D}} = \int_0^{2\pi} \frac{d\phi}{2\pi} \mathsf{D}_\phi = J_0(-iw_0\partial) . \tag{3.9}$$

At this point it is convenient to adopt the Dirac notations and to write the functions $\overline{\Phi}^+$ and Φ^- as ket-vectors $|\overline{\Phi}^+\rangle$ and $|\Phi^-\rangle$, respectively. Then, in the "coordinate" representation, we have

$$\overline{\Phi}^+(x) = \langle x|\overline{\Phi}^+\rangle , \qquad \Phi^-(x) = \langle x|\Phi^-\rangle , \tag{3.10}$$

$$\overline{\Phi}^+(x) = \int_{-\infty}^{\infty} dx' \langle x|\overline{\mathsf{D}}|x'\rangle\, \Phi^-(x') , \tag{3.11}$$

where

$$\langle x|\overline{\mathsf{D}}|x'\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp\, e^{-i(x-x')p} J_0(w_0 p) . \tag{3.12}$$

Calculating the integral we get

$$\langle x|\overline{\mathsf{D}}|x'\rangle = \overline{\Delta}^+(x|x') = \overline{\Delta}^+(y|w) = \frac{1}{\pi} \frac{\vartheta(w_0^2 - (y-w)^2)}{\sqrt{w_0^2 - (y-w)^2}} . \tag{3.13}$$

Up to now we have only been dealing with the stochasticity connected with the phase ϕ. Let us discuss what happens when we average the operator $\overline{\mathsf{D}}$ over the amplitude of the metric fluctuations. We again assume that the probability distribution is Gaussian. Using the relation

$$\int_0^\infty dz z e^{-z^2} J_0(\alpha z) = \frac{1}{2} e^{-\alpha^2/4} , \tag{3.14}$$

we get that the average of $\overline{\mathsf{D}}$ over the fluctuation amplitude is

$$< \mathsf{D} >_\omega = \exp\left(\frac{\sigma_\omega^2 \partial^2}{2}\right) . \tag{3.15}$$

The generalization of these results to a spectrum of metric fluctuations with different frequencies is straightforward. Indeed, since $< \mathsf{D} >_\omega$ has a simple exponential form, it enjoys the factorization property (2.10). Thus, using (2.14), its total ensemble average

$$\ll \mathsf{D} \gg = \exp\left(\frac{\sigma_{\text{eff}}^2 \partial^2}{2}\right) , \tag{3.16}$$

where

$$\sigma_{\text{eff}}^2 = \sum_\omega \sigma_\omega^2 . \tag{3.17}$$

Eq. (3.17) shows that the effect of the whole spectrum of metric fluctuations is to give rise to a single length which weights higher order derivative terms. This shows that the details of the fluctuations spectrum play no significant role for the operator $\ll \mathsf{D} \gg$.

The extremely simple form of the operators D in the "p"-representation allows one to make a few general observations. In particular, we have

$$\ll \Phi^+(p) \gg \equiv \langle p| \ll \Phi^+ \gg\rangle = e^{-\sigma_{\text{eff}}^2 p^2/2}\, \Phi^-(p) . \tag{3.18}$$

Thus only the high frequency components (i.e. of the order of σ_{eff}^{-1} and greater) of the initial wave packet Φ^- are strongly affected by the fluctuations of the geometry. Therefore the forward propagation of any smooth (i.e. of Fourier content much below σ_{eff}^{-1}) wave packet defined on $\mathcal{J}^-$ will not be significantly affected by the metric fluctuations. In other words, for classical black hole physics, the metric fluctuations are irrelevant if, as indicated in [6, 9], their "mean" amplitude σ_{eff} is of the order of the Planck length or smaller than it.

3.3. *Backward in time scattering*

Eventhough eq. (3.4) is perfectly symmetric in w and y, backward propagation is dramatically affected by the metric fluctuations.

To settle the discussion, we first clarify the geometrical meaning of negative y. To ease this analysis, we start with backward propagation in the absence of fluctuations. In this case, for positive y, eq. (3.4) gives $w = y$. However, since $\mathcal{J}^+$ is not a complete Cauchy surface, we need to consider the union of $\mathcal{J}^+$ and the whole event horizon $u = \infty$ in order to have a complete Cauchy surface. Thus we must introduce a coordinate along the horizon. The simplest choice consists in considering the negative values of y defined again by $y = w$. Indeed the negative y axis so defined covers the horizon from $r = 0, w = 0$ inside the collapsing shell till $w = -\infty$. Thus, the real axis $y \in (-\infty, \infty)$ forms a complete Cauchy surface and the functions $\Phi^+(y)$ determine their image $\Phi^-(w)$ on $\mathcal{J}^-$ for all w.

This procedure also applies for any given realization of the fluctuating geometry. Indeed, the event horizon, when continued backward for negative v in the inside flat geometry (2.3), reaches $r = 0$ at advanced w time $W_\phi(y = 0) = w_0 \sin(\phi + \phi_0)$. Therefore the negative half line $y \in (-\infty, 0)$ defined by eq. (3.4) still covers the whole horizon and $y \in (-\infty, \infty)$ forms a complete final Cauchy surface. Since this is valid for any realization, it is meaningful to use the coordinate $y \in (-\infty, \infty)$ after having averaged over ϕ.

For $y > 0$, $\Phi^+(y)$ gives the value on $\mathcal{J}^+$, while for $y < 0$ it gives the value on the horizon. For regular $\Phi^+(y)$, in virtue of the symmetrical role played by y and w in eq. (3.4), the averaged image on $\mathcal{J}^-$ is determined by the same scattering operator D which governed forward propagation. In the case of the full ensemble average, one has

$$\ll \Phi^- \gg = \ll \mathrm{D} \gg \Phi^+ . \tag{3.19}$$

In Fourier transform with respect to w and y this gives

$$\ll \Phi^-(p) \gg = \exp\left(-\sigma_{\text{eff}}^2 p^2/2\right) \Phi^+(p) . \tag{3.20}$$

Of special interest for studying Hawking radiation are the final images such that no incoming field emerges from the horizon for any realization of the geometry. They are of the simple form

$$\Phi^+_{out}(y) = \vartheta(y)\Phi^+(y) . \tag{3.21}$$

Unless $\Phi^+(y)$ vanish sufficiently rapidly when $y \to 0$, these modes are singular at $y = 0$.

This problem will be studied in detail in Section 5. It reveals the important role played by the fluctuating horizon geometry for backward propagation. The asymmetry between backward and forward propagation comes from the fact that the inertial time on $\mathcal{J}^+$ which characterizes out-frequencies is u and not $y = e^{-u}$. Then, the so defined out-frequencies are exponentially blue-shifted when propagated backward near the event horizon. This purely kinematical effect is at the origin of the trans-Planckian problem and has here dramatic consequences since higher derivative terms are present. Indeed, eq. (3.20) when applied to out-functions (3.21) which vanish for negative y gives, in the position representation,

$$\ll \Phi^-(w) \gg = \left[\exp\left(-\frac{1}{2} (\sigma_{\text{eff}}\, e^u \partial_u)^2 \right) \Phi^+(u) \right]_{u = -\ln w} \tag{3.22}$$

The dramatic consequences can now be seen: however smooth is the final data $\Phi^+(u)$, the fluctuations of the geometry will inevitably affect its backward propagation if it is centered around a sufficiently late retarded time. Moreover if it does not vanish sufficiently fast (i.e. faster than e^{-u}) when one approaches the horizon, the asymptotic behaviour of $\ll \mathrm{D} \gg$ intervenes. We return to these points after discussion of the properties of Hawking radiation in the fluctuating geometry.

4. Hawking radiation

The simplest way to understand why the metric fluctuations do not significantly modify the asymptotic properties of Hawking radiation is to analyse the Green function evaluated in the initial vacuum state. Indeed, as shown in [12], when the short distance expansion of this function evaluated near the event horizon reduces to the standard (Hadamard) behavior, Hawking radiation obtains on $\mathcal{J}^+$. Computation shows

$$\overline{G}^{in}(u, u') = G^{in}(u, u') + w_0^2 H(u, u') , \tag{4.1}$$

since the averaged value of all first order terms in w_0 vanishes. The crucial point is that $H(u,u')$ is finite when $u \to u'$. This implies that the corrections to Hawking radiation are at least of second order in w_0 and non-diverging in the late time regime.

Calculations give for the (quantum mechanical and statistical) mean energy flux of Hawking radiation measured at $\mathcal{J}^+$ the following expression

$$(dE/du)^{\text{pert}} = \frac{\kappa_r^2}{48\pi}\left[1+\frac{1}{2}\mu_0^2\omega^2 q^2(\omega)\right]$$

$$\equiv \frac{\kappa_r^2}{48\pi}\left[1+\mu_0^2\frac{\pi\omega}{\exp(2\pi\omega)-1}\right] \quad (4.2)$$

where

$$\kappa_r = \frac{1}{4M}\left[1+\frac{\mu_0^2}{2(1+\omega^2)}\right] \quad (4.3)$$

is the "renormalized" surface gravity (2.28).

When using the Gaussian distribution ρ_ω, the average over the amplitude $w_0(\omega)$ of the fluctuating mode gives

$$\ll\left(\frac{dE}{du}\right)^{\text{pert}}\gg = \frac{\kappa^2}{48\pi}$$

$$\times\left[1+2\sum_\omega \sigma_\omega^2\left(1+\frac{\omega^2}{2}(1+\omega^2)q^2(\omega)\right)\right]. \quad (4.4)$$

To perform this last summation requires the knowledge of the spectrum, here represented by the set of σ_ω.

The spectral distribution of the energy density flux gives more detailed information. We have

$$\frac{dE}{du\,d\lambda} = \frac{\kappa_r}{2\pi}\left[f(\Lambda)+\mu_0^2 F(\Lambda;\omega)\right], \quad (4.5)$$

where $\Lambda = \lambda/\kappa_r$,

$$f(\Lambda) = \frac{\Lambda}{\exp(2\pi\Lambda)-1}, \quad (4.6)$$

and

$$F(\Lambda;\omega) = \frac{\pi\Lambda^2 f(\omega)}{\omega^2(1+\omega^2)}$$

$$\times\left[f(\Lambda-\omega)+f(\Lambda+\omega)-2f(\Lambda)\right] \quad (4.7)$$

In order to obtain the constant part of the density of energy flux one must integrate (4.5) over the frequency λ

$$\frac{dE}{du} = \frac{\kappa_r^2}{2\pi}\int_0^\infty d\Lambda\,\left[f(\Lambda)+\mu_0^2 F(\Lambda;\omega)\right]. \quad (4.8)$$

It is easy to verify that after averaging over the spectrum (4.8) coincides with expression (4.4).

Main conclusions from these results are the following:

First, the averaged value of the outgoing flux of energy is modified. One part of this modification is connected with the renormalization of the surface gravity of the fluctuating black hole given by expression (4.3). The other part is an additional factor given by $\mu_0^2\pi\omega/(\exp(2\pi\omega)-1)$, see eq. (4.2). Both changes are second order in μ_0.

Secondly, the asymptotic spectrum of Hawking radiation is also modified. Besides the renormalization of the surface gravity which shifts the temperature, the modified spectrum (4.5) contains three additional correction terms. The two last terms in that equation contain Bose thermal factors of the form $1/(\exp(2\pi(\lambda\pm\omega)/\kappa)-1)$. In these relations, the frequency of geometry fluctuations, $\pm\omega$, plays the role of a chemical potential. The presence of such chemical potential is reminiscent to superradiance.

This fact supports the general ideas proposed by York since the appearance of these factors might be expected from the existence of a *quantum ergoshpere*. Indeed, due to quantum fluctuations, the average position of the event horizon is moved by a term proportional to the second power μ_0^2 of the amplitude of fluctuations, while the temporal position of the apparent horizon is fluctuating with amplitude μ_0. An alternative way to describe these fluctuations is to say that there exists a blurring of the physical null cone at the unperturbed horizon. Because of the existence of negative energy states inside the unperturbed black hole matter can escape from the narrow region close to the horizon. This leakage of energy is seen as Hawking radiation [6]. Under the same conditions one can expect an additional amplification of Hawking quanta while they are propagating close to the fluctuating horizon. The amplification factor we got in the expression for the modified spectrum of Hawking radiation may be considered as an indication to this effect.

5. Fluctuating geometry and trans-Planckian problem

We now discuss an analogy between the backward propagation in a fluctuating metric and the altered propagations which have been recently studied and which result from the modifications of the dispersion relation in the high frequency regime.

For this purpose, we consider the image on $v=0$ of the monochromatic wave φ_λ^{out} of out-frequency λ. The simplest way to define the action of $\ll \mathrm{D} \gg$ on our *out*-function is to work in the momentum conjugated to w. Indeed, the Fourier transform of φ_λ^{out} is well defined in the high p regime. Using eq. (3.20) we simply get

$$\ll \Phi^-_{(\lambda)}(p) \gg = \exp\left(-\frac{p^2\sigma^2_{\text{eff}}}{2}\right)\Phi^+_{(\lambda)}(p)$$
$$= \exp\left(-\frac{p^2\sigma^2_{\text{eff}}}{2}\right)\frac{\Gamma(i\lambda+1)}{\sqrt{8\pi^2\lambda}}(\epsilon+ip)^{-i\lambda-1}$$
$$= \exp\left(-\frac{p^2\sigma^2_{\text{eff}}}{2}\right)\frac{(-i)\Gamma(i\lambda+1)}{\sqrt{8\pi^2\lambda}} \times\left[e^{\pi\lambda/2}\vartheta(p)p^{-i\lambda-1} - e^{-\pi\lambda/2}\vartheta(-p)(-p)^{-i\lambda-1}\right]. \quad (5.1)$$

Since the effect of the stochastic fluctuations is to multiply the wave function by an even function in p, the relative weight encoding the Bogoliubov coefficients is unaffected. This guarantees that the vacuum state with respect to $p > 0$ leads to the usual properties of Hawking radiation, thereby proving point 2 above. Moreover, eq. (5.1) confirms that the trans-Planckian problem is tamed: The high frequency content, i.e. the near horizon behaviour, is suppressed by a Gaussian factor.

For our analysis, we determine the behaviour of $\ll \Phi^-_{(\lambda)} \gg$ in spacetime. To this end, we inverse Fourier transform separately the two terms (positive and negative p) which appear in eq. (5.1). The result is

$$\ll \Phi^-_{(\lambda)}(w) \gg = \frac{\sigma^{i\lambda}_{\text{eff}}}{\sqrt{4\pi\lambda}} Z_\lambda(w/\sigma_{\text{eff}}), \quad (5.2)$$

where

$$Z_\lambda(x) = \frac{e^{-x^2/4}}{2\sinh(\pi\lambda)} \times\left[e^{\pi\lambda/2}D_{i\lambda}(\epsilon-ix) - e^{-\pi\lambda/2}D_{i\lambda}(\epsilon+ix)\right]. \quad (5.3)$$

Here $D_\mu(z)$ is the parabolic cylinder function.

For large negative values of $x = w/\sigma_{\text{eff}}$, Z_λ vanishes. Instead, for large positive x behaves as

$$Z_\lambda(x) \sim x^{i\lambda}\left\{1 - \frac{\lambda(\lambda+i)}{2x^2} + O(x^{-4})\right\}. \quad (5.4)$$

The limit $x = w/\sigma_{\text{eff}} \to \infty$ corresponds to the far from horizon region or to $\sigma_{\text{eff}} \to 0$. The last case corresponds to no metric fluctuations. In this regime (5.4) reproduces the unperturbed out wave function $x^{i\lambda}$.

This strong dissipation is only valid for tight wave packets in λ, i.e. for $b \gg 1$. Instead, in the opposite regime $b \ll 1$, for tight wave packets in position space, the dissipation is milder. Indeed, in the limit $b \to 0$, the Gaussian factor in eq. (5.1) can be ignored in the large u_0 limit. Then, the main contribution comes from the first pole of the Γ function in the positive imaginary λ axis. In this regime the decrease of the wave packet is given by $e^{-(u_0+\ln\sigma_{\text{eff}})}$.

In brief, as long as the mean position in w on $v = 0$ of the wave packet is larger than $\lambda\sigma_{\text{eff}}$, its image is unaffected by the metric fluctuations since Z_λ still behaves as $w^{i\lambda}$. Instead, once it enters the near horizon region, its amplitude rapidly decreases.

This behavior is similar to the behavior of a backward in time propagation of a wavepacket in models with modified dispersion relation. These models have been introduced in order to show that the mutilation of the dispersion relation for frequencies higher than ω_c, which is the equivalent of σ^{-1}_{eff} in our case, in no way affect the (low energy) properties of Hawking radiation, namely stationarity and thermality. Following the original work of Unruh[13], many models have been analysed (see e.g. [14, 15, 16]). Their common property is that the Dalembertian is modified by the addition of higher derivative terms weighted by negative powers of ω_c. In these works, the modifications have been inspired by hydrodynamics[17], electrodynamics in a dielectric medium[18], field theory on a lattice theory[19], string theory[20] or by guessing what the physics near a horizon might be [21]. In all these models, the following properties obtain

1. Forwardly propagated wave packets are unaffected by the modification of the dispersion relation as long as their in-frequency content is much below the critical frequency ω_c.
2. No significant modifications of the asymptotic properties of Hawking radiation as long as the surface gravity satisfies $\kappa \ll \omega_c$.
3. Dramatic modifications of backward propagated late-time wave packets of out-frequency λ when the blue shifted value λe^u reaches ω_c.

The properties of the wavepackets propagating in a fluctuating metric possesses many similarities with these models.

6. Conclusion

It is perhaps appropriate to list what we learn from our analysis. When the relative width of the smeared horizon is small enough, i.e. when $\delta r_{EH}/r_{EH} \simeq \sigma_{\text{eff}} \ll 1$, metric fluctuations in the near horizon geometry affect the asymptotic properties of Hawking radiation only slightly, in the second order of σ_{eff}. The reason for this stability can be seen from the short distance behaviour of the *in*-Green function. Backward propagated wave packets representing Hawking quanta of energy λ are dissipated when their Doppler shift frequency $i\partial_r$ reaches σ_{eff}^{-1}, i.e. when their separation in r from the event horizon approaches $\lambda\sigma_{\text{eff}}$.

Why the high frequency behavior of the in-Green function differs so much from that of backward scattered waves? The reason is the following. Being a function of the difference in V_ϕ, the in-Green function is hardly sensitive to the metric fluctuations in the coincidence point limit. On the contrary backscattered wave functions defined on $\mathcal{J}^+$ are sensitive to the metric fluctuations they have encountered when evaluated near the horizon. Moreover, since their frequency is blue shifted, they are inevitably strongly affected by the metric fluctuations.

In brief, the main outcome of the paper is to have provided physical foundations in terms of metric fluctuations to the concept of effective propagation of light near a black hole horizon.

This allows to address in a rational scheme the question of the domain of validity of this effective propagation. It also provides an explanation for the vexing question of the apparent violation of local Lorentz invariance[14, 15, 21]. The neatest way to characterize this violation is to focus on the near horizon behavior of a monochromatic mode ϕ_λ^{out}. In the absence of modification of the dispersion relation, this mode behaves as $w^{i\lambda}$ where $w = 2r - 1$. Hence there is no length which allows one to distinguish low from high momenta. This absence is a consequence of the local Lorentz invariance of theories based on the usual Dalembertian. On the contrary, when dealing with a modified dispersion relation, one breaks this invariance since the new dynamical equation is written in a preferred frame. For acoustic black holes this makes good sense since both the frame and the critical length, which characterizes what "high" frequency means, are given by the constituents of the fluid. On the contrary, it is rather unclear to see the origin of such a preferred frame for a gravitational black hole. One of the main virtues of the present work is to provides a simple answer to this puzzle. Indeed the ensemble of metric fluctuations unambiguously determines, σ, the constant spread in r (measured along $v = 0$) of the distribution of the backward propagated rays representing the event horizon. Because of the hypothesis of stationarity metric fluctuations, the modified equation governing light propagation has a simple and stationary expression in the v, r coordinate system. In particular, the cut-off length σ appears only through powers of $\sigma\partial_r|_v$. In this case what might be interpreted as the origin of a "violation of Lorentz invariance" originates from the ensemble of stationary metric fluctuations.

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КВАНТОВЫЕ ЭФФЕКТЫ ВО ФЛУКТУИРУЮЩЕЙ ГЕОМЕТРИИ ЧЕРНОЙ ДЫРЫ

С. Барабье, В. Фролов

Мы изучаем распространение световых лучей и безмассовых полей во флуктуирующей геометрии черной дыры. Флуктуации метрики индуцируются малым осциллирующим световым падающим потоком энергии, описываемым посредством статистического ансамбля. Стохастическими переменными являются фазы и амплитуды его Фурье-мод. Интегрируя по этим переменным, мы находим, что эффективное распространение поля обладает следующими свойствами. Амплитуда флуктуаций метрики определяет некоторую критическую длину: пакеты, являющиеся по сравнению с ней гладкими, не подвергаются существенному воздействию при распространении вперед во времени. Соответственно, мы находим, что асимптотические свойства излучения Хокинга изменяются незначительно. Однако волновые пакеты, распространяющиеся вспять во времени, испытывают сильное воздействие со стороны флуктуаций метрики, когда их частота, смещаясь в голубую сторону, достигает обратной критической длины.

КВАНТОВІ ЕФЕКТИ У ФЛУКТУЮЮЧІЙ ГЕОМЕТРІЇ ЧОРНОЇ ДІРИ

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Ми вивчаємо поширення променів світла та безмасових полів у флуктуюючій геометрії чорної діри. Флуктуації метрики породжуються малим осцилюючим падаючим світловим потоком енергії, що описується як статистичний ансамбль. Стохастичними змінними є фази та амплітуди його Фур'є-мод. Інтегруючи по цих змінних ми знаходимо, що ефективне поширення поля має такі властивості. Амплітуда флуктуацій метрики породжує деяку критичну довжину: пакети, які є у порівнянні з нею гладкими, не зазнають істотного впливу при поширенні вперед у часі. Відповідно до цього ми знаходимо, що асимптотичні властивості випромінювання Хокінга змінюються мало. Однак хвильові пакети, що поширюються навпроти часу, зазнають значного впливу з боку флуктуацій метрики, коли їх частота, зміщуючись до блакитної частини спектру, досягає зворотної критичної довжини.