Radio Physcics and Radio Astronomy, 2001, v. 6, No. 2, pp. S53–S56

T-SOLUTIONS FOR ANISOTROPIC FLUID SPHERES IN GENERAL RELATIVITY

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A number of exact solutions of the Einstein equations for nonstatic spherically symmetric configurations of an anisotropic fluid are derived and investigated. The metric coefficient $g_{\theta\theta}$ is assumed to be the function of the time coordinate only (T-solutions). The configurations under consideration are shear-free. Equations of state for the radial and tangential components of pressure are chosen in various forms.

1. Introduction

We consider exact solutions of the Einstein field equations for a spherically symmetric distributions of matter. The component $g_{\theta\theta}$ of the metric tensor is assumed to be depending only on the time coordinate t. Such solutions describe only the T-regions of space-time and are called "T-solutions" following Novikov. The T-solutions are also known as solutions of the Kantovski - Sachs type. They play a significant role for the investigation of the later stages of a gravitational collapse and also for studying the early Universe. The first T-solution was derived by Novikov. It describes the spherically symmetric empty space. Then Kantowski and Sachs [1] obtained T-solution for the dust matter. The T-solutions for a perfect fluid spheres were derived later (see [2-6]). For the configurations of spherically symmetric perfect fluid with nonzero pressure it is known two types of the T-solutions:

1) Parabolic type (the fluid sphere unlimited expands or collapses);

2) Elliptic type (pulsating model that expands from initial singularity to the maximal radius and then collapses).

However, it was found that the presence of electromagnetic fields may significantly changes the type of the solution. The collapse of T-regions for the Reissner - Nordstrom solution may be stopped. Thus the bouncing solutions we will call the solutions of hyperbolic type. The number of works considering solutions with equation of state of an anisotropic fluid increases greatly in last decades. The energy-momentum tensor for the spherically symmetric anisotropic matter $T_1^1 \neq T_2^2$ in comoving coordinates. From physical point of view the appearance of a local anisotropy may be caused by scalar field, neutrino radiation, existence of a solid core etc. Besides, the energy-momentum tensor of two perfect fluids moving with different velocities can be transformed into the form of the anisotropic fluid [7].

Different solutions for spherically symmetric configurations of anisotropic fluid were derived and investigated (see [7, 8] and referenced works). An astrophysical models based on this solutions were discussed, and it was shown that some properties of such an anisotropic models may differ drastically from the properties of the isotropic ones. However, as authors know, the anisotropic T-solutions were not considered.

So, the aim of the present paper is to obtain and investigate the exact T-solutions for spherically symmetric anisotropic fluid and to compare the anisotropic models with the isotropic ones.

2. Field equations

Consider a spherically symmetric configuration of the anisotropic fluid. We assume that the component $g_{\theta\theta} = r^2$ of the metric tensor depends only on time coordinate t, and shear is zero. Thus the line element in comoving frame can be written as

$$ds^2 = dt^2 - r^2(t)(dR^2 + d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (2.1)$$

where r is scale factor, R and t are radial and time coordinates, θ and φ are spherical angles. The energy-momentum tensor of the anisotropic fluid has diagonal form: $T_0^0 = \varepsilon$, $T_1^1 = -p_r$, $T_2^2 = T_3^3 = -p_\perp$, where ε is the energy density, p_r and $p_\perp$ are "radial" and "tangential" pressure, respectively.

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The Einstein field equations for the configurations under consideration take the form

$$8\pi\varepsilon = (3\dot{r}^2+1)/r^2,\ -8\pi p_r = (2\ddot{r}+\dot{r}^2+1)/r^2,$$

$$-8\pi p_\perp = (2\ddot{r}+\dot{r}^2)/r^2, \quad (2.2)$$

where dots denote differentiation with respect to t.

The difference between tangential and radial components of the pressure $\Delta p = 8\pi(p_\perp - p_r) = r^{-2} > 0$. From this equation we can see that where is no shear-free T-solution for a perfect fluid sphere. T-solution for empty space also has nonzero shear and don't belong to the considered class.

Consider T-solutions with various equations of state.

3. Models with $p_r \sim \varepsilon$

Assume that the radial pressure is $p_r = n\varepsilon$, $n = Const$. Then from field equations (2.2) we have

$$2\frac{\ddot{r}}{r} + (3n+1)\left(\frac{\dot{r}}{r}\right)^2 + (n+1)\frac{1}{r^2} = 0. \quad (3.1)$$

By solving this equation, for the case $n \neq -1/3$ we find

$$\int\left(Ar^{-3n-1} - \frac{n+1}{3n+1}\right)^{-1/2} dr = \pm(t-t_0), \quad (3.2)$$

where A and t_0 are arbitrary constants.

In the case $n = -1/3$ (this is anisotropic analogue of the Einstein's Universe with $p_r = -\varepsilon/3$) eq. (3.1) gives

$$\dot{r} = \pm\sqrt{\frac{2}{3}\ln(r/r_0)},\ 8\pi\varepsilon = \frac{1-2\ln(r/r_0)}{r^2},$$

$$8\pi p_\perp = 2\frac{\ln(r/r_0)+1}{3r^2}, \quad (3.3)$$

where $r_0 = Const$, $r \leq r_0$. Integrating (3.3), we obtain

$$r = r_0 e^{-x^2/2};\ \pm(t-t_0) = \sqrt{3}r_0\int_0^x e^{-\xi^2/2}d\xi, \quad (3.4)$$

where $x = r/r_0$, $t_0 = Const$. The configuration expands to the maximal radius $r = r_0$, and then collapses (elliptic type). The energy density is positive, $p_r < 0$, and tangential pressure is negative for $r < r_0/e$.

In the case when the constant A in eq. (3.2) is equal to zero, the solution is

$$r = \pm\alpha(t-t_0),\ 8\pi\varepsilon = (3\alpha^2+1)r^{-2},$$

$$8\pi p_r = -(1+\alpha^2)r^{-2},\ 8\pi p_\perp = -\alpha^2 r^{-2}, \quad (3.5)$$

where $\alpha = \sqrt{-(n+1)/(3n+1)}$. Such models are possible only if $-1 < n < -1/3$. The radius r is proportional to time coordinate t, and both p_r and $p_\perp$ are negative. The model is of parabolic type.

Equation (3.2) also may be easily integrated for cases $n = -1$, $n = 1/3$, $n = -2/3$ and $n = 0$ ($p_r = 0$).

3.1. For a model with $n = -1$ (anisotropic analogue of De Sitter Universe with $p_r = -\varepsilon$) we find $r = r_0\exp(\pm At)$, where r_0 and A are constants ($A \neq 0$, $r_0 \neq 0$). The model is of a parabolic type. The radius depends on time by exponential law, and $\varepsilon > 0$, $p_r < 0$, $p_\perp = -3A^2 = Const$, $p_\perp < 0$, i.e. tangential pressure is constant. On $r \to \infty$: $\varepsilon \to -p_\perp$, $p_r \to p_\perp$. On $r \to 0$: $\varepsilon \to \infty$, $p_r \to -\infty$.

The solution with $n = -1$ can be also written as:

$$r = r_0\exp(\pm t\sqrt{-8\pi p_\perp/3}),$$

$$\varepsilon = -p_\perp + \frac{1}{8\pi r^2},\ p_r = -\varepsilon,\ p_\perp = Const. \quad (3.6)$$

3.2. For $n = 1/3$ (analogue of the ultrarelativistic equation of state for anisotropic fluid) we get:

$$r = \sqrt{r_m^2 - 2t^2/3},\ 8\pi\varepsilon = (2r_m^2 - r^2)r^{-4},$$

$$p_r = \frac{1}{3}\varepsilon,\ p_\perp = \frac{1}{3}\varepsilon + \frac{1}{8\pi r^2}, \quad (3.7)$$

where $r_m > 0$ is a parameter. The time coordinate t must satisfy the inequality $-r_m \leq \sqrt{2/3}t \leq r_m$. The model expands from initial singularity $r \to 0$ to the maximal radius $r = r_m$ ($t = 0$), and then collapses (elliptic type). The scale factor $r \to 0$ if $t \to \pm r_m$. The energy density is positive, for r_m it is $8\pi\varepsilon = r_m^{-2}$, for $r \to 0$: $\varepsilon \to \infty$. Pressures $p_r \to \infty$, $p_\perp \to \infty$ if $r \to 0$.

3.3. Consider case $n = -2/3$. Integrating (3.2) obtain

$$r = \frac{C}{4}t^2 - \frac{1}{3C},\ 8\pi\varepsilon = \frac{3Cr+2}{r^2},$$

$$p_r = -\frac{2}{3}\varepsilon,\ 8\pi p_\perp = -\frac{2Cr+1/3}{r^2}, \quad (3.8)$$

where $C \neq 0$ is constant. For $C < 0$ the solution is of elliptic type, the maximal value of $r(t)$ is $r_{max} = -1/(3C)$, the energy density and pressures are $\varepsilon = 9C^2$, $p_\perp = 3C^2$. In the case $C = 0$ the solution transforms into (3.5) with $n = -2/3$.

4. Models with $p_\perp \sim \varepsilon$

Consider solutions with the equation of state $p_\perp = \gamma\varepsilon$, $\gamma = Const$. From the field equations (2.2) we get:

$$2r\ddot{r} + (3\gamma+1)\dot{r}^2 + \gamma = 0. \quad (4.1)$$

The general solution of this equation for $\gamma \neq -1/3$ is of the form

$$\int \left(Dr^{-1-3\gamma} - \frac{\gamma}{3\gamma+1} \right)^{-1/2} dr = \pm(t-t_0), \quad (4.2)$$

where $D \neq 0$ and t_0 are constants. In case $D = 0$ the solution transforms into form of eq.(3.5) with $\alpha = \sqrt{-\gamma/(3\gamma+1)}$.

For $\gamma = -1/3$ we get $\dot{r} = \pm\sqrt{3^{-1}\ln(r/r_0)}$, $r_0 = Const$.

Equation (4.2) may be easily integrated for the cases $\gamma = 1/3$, $\gamma = -2/3$, $\gamma = -1$ and $\gamma = 0$.

4.1. For $\gamma = 1/3$ (an anisotropic analogue of ultrarelativistic equation of state) integrating (4.2) we get

$$r = \sqrt{6D - (t-t_0)^2/6}, \; 8\pi\varepsilon = \frac{6D - r^2}{2r^4} + \frac{1}{r^2},$$

$$p_\perp = \frac{1}{3}\varepsilon, \; p_r = p_\perp - \frac{1}{8\pi r^2}. \quad (4.3)$$

where $t_0 = Const$, $D > 0$. The model expands from the initial singularity to $r_{max} = \sqrt{6D}$, and then collapses, $r \to 0$ if $t \to t_0 \pm 6\sqrt{D}$ (elliptic type). The energy density and tangential pressure are positive, $p_r < 0$ for $r > \sqrt{6D/5} = r_{max}/\sqrt{5}$.

4.2. For the model with $\gamma = -2/3$ we obtain

$$r = \frac{2}{3D} + \frac{D}{4}t^2, \; 8\pi\varepsilon = \frac{3Dr}{r^2} \frac{1}{},$$

$$p_\perp = -\frac{2}{3}\varepsilon, \; 8\pi p_r = \frac{6Dr+1}{3r^2} \quad (4.4)$$

where $D > 0$ is a parameter. The configuration contracts from infinity to minimal radius $r_{min} = 2/(3D)$, and then expands to the initial state. Hence, this solution is of a hyperbolic type.

The energy density is positive, $\varepsilon \to 0$, $p_r \to 0$ if $t \to \pm\infty$. Radial and tangential pressures are negative, $8\pi\varepsilon \to 9D^2/4$, $8\pi p_r \to -15D^2/4$ for $r \to r_{max}$.

4.3. In case $\gamma = -1$ (anisotropic analogue of De Sitter model with $p_\perp = -\varepsilon$) the solution of eq. (4.2) is

$$r = \frac{1}{\sqrt{2D}}\cosh[\sqrt{D}(t-t_0)], \; 8\pi\varepsilon = 3D - \frac{1}{2r^2},$$

$$p_\perp = -\varepsilon, \; 8\pi p_r = -3D - \frac{1}{2r^2}, \quad (4.5)$$

where $t_0 = Const$, $D > 0$. The model is of hyperbolic type again: $r \to r_{\min} = 1/\sqrt{2D}$ (for $t \to t_0$), $\varepsilon \to D/(4\pi)$, $p_r \to -D/(2\pi)$.

5. Models with $p_r = Const$

In the case $p_r = Const$ the solution of the Einstein equations (2.2) takes the form

$$\dot{r} = \pm\sqrt{\frac{A}{r} - \frac{8\pi p_r}{3}r^2 - 1},$$

$$\varepsilon = -p_r + \frac{3A - 2r}{8\pi r^3}, \; p_\perp = p_r + \frac{1}{8\pi r^2}, \quad (5.1)$$

where A is an arbitrary constant.

For the models with zero radial pressure ($p_r = 0$) we obtain

$$r = A\sin^2\frac{\chi}{2}, \; \pm(t-t_0) = \frac{A}{2}(\chi - \sin\chi). \quad (5.2)$$

Configuration expands to the maximal radius $r = A$, and then collapses (elliptic type). The energy density and tangential pressure are positive.

Putting $A = 0$ in eq. (5.1) we get a hyperbolic solution:

$$r = r_0\cosh\frac{t-t_0}{r_0}, r_0 = \sqrt{-\frac{3}{8\pi p_r}}. \quad (5.3)$$

This model contracts to the minimal radius $r = r_0$, and then expands. Both radial and tangential pressures are negative.

6. Models with $p_\perp = Const$

For models with a constant tangential pressure we get from (2.2)

$$\dot{r} = \pm\sqrt{\frac{B}{r} - \frac{8\pi p_\perp}{3}r^2},$$

$$\varepsilon = -p_\perp + \frac{3B + r}{8\pi r^3}, p_r = p_\perp - \frac{1}{8\pi r^2}, \quad (6.1)$$

where $B = Const$.

In specific case $p_\perp = 0$ the solution takes the form

$$r = [3\sqrt{B}(t-t_0)/2]^{2/3}, \; t_0 = Const. \quad (6.2)$$

So, the model with zero tangential pressure unlimited expands or collapses (parabolic type). The radial pressure is negative.

The solution (6.1) with $B = 0$ takes the form (3.6) (solution with $p_r = -\varepsilon$).

7. Models with $p_\perp = kp_r$

Assuming the equation of state to be of the form $p_\perp = kp_r$, $k \neq 1$ (a perfect fluid is not included in considered class), we find from (2.2)

$$\dot{r} = \pm\sqrt{\frac{C}{r} + \frac{k}{1-k}}, \; 8\pi\varepsilon = \frac{3C}{r^3} + \frac{2k+1}{(1-k)r^2},$$

$$p_r = \frac{1}{8\pi(k-1)r^2}, \quad (7.1)$$

where $C = Const$.

Depending on sign of C and $k/(1-k)$ we have five different solutions.

7.1. For $C = 0$ and $0 < k < 1$ we have solution (3.5) with $\alpha = \sqrt{k/(1-k)}$.

7.2. For $k = 0$ we have solution (5.2) with zero tangential pressure.

7.3. For $C < 0$ and $0 < k < 1$ we get a hyperbolic model:

$$r = |C|\frac{1-k}{k}\cosh^2\frac{\chi}{2},$$

$$\pm(t-t_0) = \frac{|C|}{2}\left(\sqrt{\frac{1-k}{k}}\right)^3(\sinh\chi + \chi). \quad (7.2)$$

The pressures are negative.

7.4. For $C > 0$ and $0 < k < 1$ we have a parabolic model:

$$r = C\frac{1-k}{k}\sinh^2\frac{\chi}{2},$$

$$\pm(t-t_0) = \frac{C}{2}\left(\sqrt{\frac{1-k}{k}}\right)^3(\sinh\chi - \chi). \quad (7.3)$$

The pressures are negative.

7.5. For $C > 0$ and $k < 0$ or $k > 1$ we get an elliptic model:

$$r = C\frac{k-1}{k}\sin^2\frac{\chi}{2},$$

$$\pm(t-t_0) = \frac{C}{2}\left(\sqrt{\frac{k-1}{k}}\right)^3(\chi - \sin\chi). \quad (7.4)$$

The pressures are positive if $k > 1$. For $k \to \infty$ this solution transforms to the solution (5.2) with zero radial pressure.

Also it can be shown that the solution with equation of state $\Delta p \sim \varepsilon$ has the form (3.5).

8. Conclusions

New exact T-solutions of the Einstein equations for nonstatic spherically symmetric configurations of shear-free anisotropic fluid are derived and investigated. It is shown that presence of a local anisotropy of pressure may lead to significant changes in the evolution of the model. For perfect fluid spheres all known solutions are either of a parabolic or of an elliptic type. For the anisotropic fluid besides this the following types are possible: $\exp(t)$; 3) relatively slow dependence of scale factor on t $(r \sim t)$.

For some models the pressure is always negative. Such a solutions in general relativity may be applied to the classical description of the particle-production phases in early Universe predicted by some symmetry-breaking particle theories [4].

The obtained solutions describe -regions of space-time. They could have cosmological application and be used for description of processes in the early Universe.

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Т-РЕШЕНИЕ ДЛЯ ЖИДКИХ АНИЗОТРОПНЫХ СФЕР В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ

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Получен и исследован ряд точных решений уравнений Эйнштейна для нестатической сферически-симметричной анизотропной жидкости. Метрический коэффициент $g_{\theta\theta}$ считается зависящим только от временной координаты (Т-решения). Рассматриваемые конфигурации имеют равный нулю сдвиг. Построены модели с различными уравнениями состояния для радиальной и тангенциальной составляющих давления.

Т-РІШЕННЯ ДЛЯ РІДКИХ АНІЗОТРОПНИХ СФЕР У ЗАГАЛЬНІЙ ТЕОРІЇ ВІДНОСНОСТІ

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Одержаний та досліджений ряд точних рішень рівнянь Ейнштейна для нестатичної сферично-симетричної анізотропної рідини. Метричний коефіцієнт $g_{\theta\theta}$ вважається таким, що залежить тільки від часової координати (Т-рішення). Конфігурації, які розглядаються, мають нульовий зсув. Побудовані моделі з різними рівняннями стану для радіальної та тангенціальної складових тиску.