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Analysis of Tridiagonal Recurrence Relations in Continuum Approximation

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Transition from difference to differential equation allows solving tridiagonal recurrence relations, which appear, among other things, in analysis of the rotation of an overdamped Brownian particle subjected to a periodic force. Replacement of the discrete integers in the Fourier series by continuum is justified for large numbers, i. e. for small angles. For the simplest case of the sinusoidal force, our solution, indeed, coincides with one obtained by expanding the sin in the original Fokker-Planck equation (The Ornstein-Uhlenbeck limit). However, for slightly more complicate potential the expansion for small angles does not transform the appropriate Fokker-Planck equation into the soluble. At the same time, the method suggested allows solving the problem for all periodic potentials which have finite number of terms in their Fourier series such as $\sin^m(\theta)$ or $\cos^m(\theta)$. Even and odd functions require slightly different analysis, and are considered separately.

The current radio physics is essentially nonlinear science. This is caused by using high power generators being concerned with their arrangement as well as with the effect of the radiation produced on various objects. Nonlinear processes occur in the natural conditions too.

Now, one can speak with assurance about the occurence in Kharkov of the lead of nonlinear radio physics which found the world recognition. The first from the papers in which the complicated nonlinear problem was solved, and which can be called the basic one, was the theory of magnetron built by S. Ya. Braude. He succeeded in solving the system of nonlinear equations describing the electron motion in magnetron. This work as is known was approved by L. D. Landau.

In present paper the new method is proposed for solving the nonlinear problems describing the radio physical phenomena including the behavior of the nonlinear circuit with strong resistivity, the motion of an electron in nonlinear resistive medium, et cetera.

The present problem is related to the same field as the pioneer work of S. Ya. Braude.

1. Introduction

Many ordinary and partial differential equations used in a practice after suitable expansion reduce to tridiagonal recurrence relation of the form

$$\frac{\mathrm{d}c_n}{\mathrm{d}t} = Q_n^- c_{n-1} + Q_n c_n + Q_n^+ c_{n+} \qquad (1)$$

Some examples are listed in the Risken monograph [1] (master equation with nearest-neighbor coupling, the one – dimensional Shroedinger equation with an unharmonic potential, the Fokker – Plank equation for lasers, and for the Brownian particle moving in a periodic potential). The unknown quantities c_n in Eq. (1) might be scalars or column vectors. We restrict our analysis to the case of scalars; the extension to vectors should present no problems.

Overdamped Brownian motion in a periodic potential is a typical example leading to Eq. (1). Its equation of motion has the following form:

$$\frac{d\theta}{dt} + g(\theta) = a + f(t), \qquad (2)$$

where $g(\theta)$ and a are the periodic and constant forces acting on a particle, in addition $< f(t) f(t_1) >= 2D\delta(t - t_1)$.

The Fokker-Planck equation for the distribution function $P(\theta, t)$, which corresponds to the Langevin equation (2), has the following form:

$$\frac{dP}{dt} = \frac{\partial}{\partial\theta}\left[\left(g(\theta) - a\right)P\right] + D\frac{\partial^2 P}{\partial\theta^2}. \qquad (3)$$

If Eq. (3) has coefficients periodic in θ, then its solution is also periodic in θ, and, therefore, may be expanded in the Fourier series

$$P(\theta, t) = \sum_{n=-\infty}^{n=\infty} c_n(t)\exp(in\theta). \qquad (4)$$

Substituting the expansion of the periodic function $g(\theta)$ in the Fourier series

$$g(\theta) = \sum_{n=-\infty}^{n=\infty} d_k \exp(ik\theta) \qquad (5)$$

and (4) into (3), one obtains after simple transformation

$$\frac{dc_n}{dt} = (-Dn^2 - ina)c_n - in\sum_{k=-\infty}^{k=\infty} d_k c_{n-k}. \qquad (6)$$

If the sum in Eq. (6) contains only a finite number of terms or if this sum converges so rapidly that one can restrict our analysis to a finite number of terms, then Eq. (6) takes the form of scalar (or vector) tridiagonal recurrence relations (1) [1].

The equation of motion of an overdamped pendulum subject to a constant and random torque is the simplest example of an equation of the form (2) (with $g(\theta) = \sin\theta$):

$$\frac{d\theta}{dt} + b\sin\theta = a + f(t). \qquad (7)$$

Eq. (7) describes many different phenomena, such as motion of fluxons in superconductors [2], motion of ions in superionic conductors [1] and biological channels [3], charge density waves [4], phase locking in electric circuits [3], mode locking in ring laser gyroscopes [5], and the Josephson junction [6].

The Fokker-Planck equation corresponding to Eq. (7) has the form (3) (with $g(\theta) = \sin\theta$):

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial\theta}(\sin\theta - a)P + D\frac{\partial^2 P}{\partial\theta^2}, \qquad (8)$$

or the form (6) with the coefficients $d_1 = -d_{-1} = \dfrac{1}{2i}$ and $d_k = 0$ for $k \neq 1$:

$$\frac{dc_n}{dt} = (-Dn^2 - ina)c_n - \frac{bn}{2}(c_{n+1} - c_{n-1}). \qquad (9)$$

The usual way to solve the tridiagonal recurrence relations of type (1) is by matrix continued fractions (see [1] and references therein). The aim of this note is to propose another method for solving Eqs. (1), (9), namely, the transition from a difference to a differential equation. Of course, this method is not new, and it has been used successfully, for example, in continuum approximation of small oscillations near the equilibrium positions for chains of equidistant particles with nearest-neighbor interactions. However, this method has never been applied to the equations of a kind (9), which would be the aim of our considerations.

Replacing the integers n in Eq. (9) by the continuous variable, one can make the Taylor expansion which gives

$$\left(c_{n+1} - c_{n-1}\right) \cong 2\frac{\partial c_n}{\partial n} + O\left(\frac{2}{3}\frac{\partial^3 c_n}{\partial n^3}\right). \qquad (10)$$

The error due to retaining only the first term in the series expansion in (10) is of order of $\frac{2}{3}\frac{\partial^3 c_n}{\partial n^3}\left(2\frac{\partial c_n}{\partial n}\right)^{-1} \approx \frac{1}{3n^2}$, which strongly decreases with *n*.

Substituting (10) into (9), one gets

$$\frac{\partial c_n}{\partial t} + bn\frac{\partial c_n}{\partial n} = [-Dn^2 - ina]c_n. \qquad (11)$$

Let us now turn to the general Eq. (6), assuming that summation in this equation is restricted to some $k_{\max} < \infty$. Then, for $n > k_{\max}$, one can expand c_{n-k} in Eq. (6), $c_{n-k} \approx c_n - k\frac{\partial c_n}{\partial n} + \frac{k^2}{2}\frac{\partial^2 c_n}{\partial n^2} + ...,$ which reduces Eq. (6) to the following form

$$\frac{\partial c_n}{\partial t} = \left[-Dn^2 - ina - in\sum_{k=-k_{\max}}^{k_{\max}} d_k\right]c_n - in\sum_{k=-k_{\max}}^{k_{\max}}(kd_k)\frac{\partial c_n}{\partial n} + \frac{in}{2}\sum_{k=-k_{\max}}^{k_{\max}}(k^2 d_k)\frac{\partial^2 c_n}{\partial n^2}. \qquad (12)$$

Three combination of the Fourier coefficients d_k defined by (5), enter Eq. (12), $K_0 = \sum_{k=-k_{\max}}^{k_{\max}} d_k$, $K_1 = \sum_{k=-k_{\max}}^{k_{\max}} kd_k$, and $K_2 = \sum_{k=-k_{\max}}^{k_{\max}} k^2 d_k$. The calculations are slightly different for the odd and even function $g(\theta)$. For odd functions $g(\theta)$, $d_{-k} = -d_k$ so that $K_0 = 0$ and $K_1 = 2\sum_{k=0}^{k_{\max}} kd_k$, whereas for even functions $g(\theta)$, $K_0 = 2\sum_{k=0}^{k_{\max}} d_k$ and $K_1 = 0$. Therefore, for odd $g(\theta)$, one can neglect the last term in Eq. (12), and rewrite it as

$$\frac{\partial c_n}{\partial t} = \left[-Dn^2 - ina\right]c_n - inK_1\frac{\partial c_n}{\partial n}. \qquad (13)$$

Whereas for the even function $g(\theta)$

$$\frac{\partial c_n}{\partial t} = \left[-Dn^2 - ina - inK_0\right]c_n + \frac{in}{2}K_2\frac{\partial^2 c_n}{\partial n^2}. \qquad (14)$$

Analogous to Eq. (10), one concludes that the relative error due to neglecting the next term in the expansion in Eq. (14) is of the order

$$\frac{1}{24}\frac{\partial^4 c_n}{\partial n^4}\left(\frac{1}{2}\frac{\partial^2 c_n}{\partial n^2}\right)^{-1} \approx \frac{1}{12n^2}.$$

We consider in Sections 2 and 3 the simple cases of purely deterministic and steady-state overdamped pendulum, and compare our result with the exact solutions, leaving to Section 4 the analysis of (11). The general analysis of even and odd functions $f(\theta)$ described by Eqs. (13) and (14) is performed in Sections 5 and 6. Finally, some discussion and conclusions complete the analysis.

2. Non-biased pendulum

Let us start with the simplest case, without deterministic (a) or random ($f(t)$) forces in Eq. (7). Then, Eq. (7) takes the simple form, $\frac{d\theta}{dt} + b\sin\theta = 0,$ which allows the exact solution

$$\tan\frac{\theta}{2} = \tan\frac{\theta_0}{2}\exp(-bt), \qquad (15)$$

where $\theta(t=0) = \theta_0$.

Eq. (8) with $a = D = 0$ transforms after inserting $G = P\sin\theta$ to a first order partial differential equation with constant coefficients and characteristic equations of the form

$$\frac{dt}{1} = \frac{d\theta}{b\sin\theta} = \frac{dG}{0}. \tag{16}$$

The latter equations show that t and θ enter the solution only in the combination $bt + \log\left(\tan\frac{\theta}{2}\right)$, so that the solution of Eq. (8) with $a = D = 0$ has the following form:

$$P = \frac{f\left[bt + \log\left(\tan\frac{\theta}{2}\right)\right]}{\sin\theta}, \tag{17}$$

where $f(z)$ is an arbitrary function which is found from the initial conditions.

If $P(\theta, t=0) = \delta(\theta - \theta_0)$, then Eq. (17) gives

$$\delta(\theta - \theta_0) = \frac{f\left[\log\left(\tan\frac{\theta}{2}\right)\right]}{\sin\theta}, \tag{18}$$

which means that

$$f(z) = \sin\left[2\tan^{-1}(\exp z)\right]\delta\left[2\tan^{-1}\left(\frac{\theta}{2}\exp z\right) - \theta_0\right]. \tag{19}$$

Substituting now (19) at value $z = bt = \log\left(\tan\frac{\theta}{2}\right)$ into (17), one finds the coefficients c_n in the Fourier series (4),

$$c_n = \int d\theta \exp(-in\theta)\frac{\sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\exp(bt)\right)\right]}{\sin\theta}\times$$

$$\delta\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\exp(bt)\right) - \theta_0\right]. \tag{20}$$

Using the well-known relation,

$$\delta[\psi(\theta) - \theta_0] = \left[\frac{d\psi}{d\theta}(\theta = \tilde{\theta})\right]^{-1}\delta(\theta - \tilde{\theta})$$

with $\psi(\tilde{\theta}) = \theta_0$ one can perform integration in Eq. (20) which finally gives

$$P(\theta, t) = \frac{\sin\theta_0}{\sin\left[2\tan^{-1}\left(\tan\frac{\theta_0}{2}\exp(-bt)\right)\right]}\times$$

$$\frac{\left[1 + \tan^2\left(\frac{\theta_0}{2}\right)\right]\exp(-bt)}{1 + \tan^2\left(\frac{\theta_0}{2}\right)\exp(-2bt)}\times$$

$$\sum_n \exp in\left[\theta - 2\tan^{-1}\left(\tan\frac{\theta_0}{2}\exp(-bt)\right)\right]. \tag{21}$$

The sum over n in the exponent of the latter formula defines the delta-function which gives the solution (15) while the pre-sum normalization factor equals to unity both for $t = 0$ and $t = \infty$. Hence, using the more complicated calculation, we obtained the same solution, which follows immediately from the equation of motion, as it should be since our calculation is exact.

Let us now solve the approximate Eq. (11) and, then, compare the result obtained with the exact one found both from the equation of motion and from the Fokker-Planck equation.

The first-order partial differential Eq. (11) with constant coefficients has the characteristic equations of the form

$$\frac{dt}{1} = -\frac{dn}{bn} = \frac{dc_n}{0}. \tag{22}$$

The latter equation is similar to Eq. (16), and its solution can be found analogously to Eqs. (16)-(21), which gives

$$P(\theta,t)=\sum_n \exp[in(\theta-\theta_0\exp(-bt))] \qquad (23)$$

with the deterministic solution $\theta=\theta_0\exp(-bt)$ which is the limiting case of the exact solution (15) for small θ.

3. Steady-state case

In the steady-state limit, $\dfrac{\partial c_n}{\partial t}=0,$ Eq. (9) reduces to an ordinary equation in finite differences

$$(-Dn^2-ina)c_n-\frac{bn}{2}(c_{n+1}-c_{n-1})=0, \qquad (24)$$

which can be solved rigorously. The continued-fraction method was used for the solution of Eq. (24) by Cresser et al. [7], while Ivanchenko and Zilberman [8] noticed that Eq. (24) bears a resemblance to the recurrence relation for the Bessel functions of the imaginary argument [9], $2\nu I_\nu(z)+z[I_{\nu+1}(z)-I_{\nu-1}(z)]=0.$ Comparing the latter equation with Eq. (24), one concludes that (up to a normalized factor),

$$c_n\approx I_{n+\frac{ia}{D}}\left(\frac{b}{D}\right). \qquad (25)$$

For the following discussion, we need the asymptotic form of Eq. (25) for large numbers $n+ia/D$. The latter can be obtained from the integral representation of this function [9],

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi \exp(z\cos\theta)\cos(\nu\theta)\mathrm{d}\theta-$$

$$\frac{\sin(\nu\pi)}{\pi}\int_0^\infty \exp(-z\cosh t-\nu t)\mathrm{d}t. \qquad (26)$$

For large ν one can neglect the second integral in (26), and the main contribution to the first integral comes from small θ since for large θ the kernel of this integral oscillates rapidly. In line with this, one can extend the range of integration to infinity and expand the argument $\cos(\theta)\approx 1-\theta^2/2.$ Then,

$$I_\nu(z)=\frac{\exp(z)}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{z\theta^2}{2}\right)\cos(\nu\theta)\mathrm{d}\theta=$$

$$\frac{\exp(z)}{\sqrt{2\pi z}}\exp\left(-\frac{\nu^2}{2z}\right) \qquad (27)$$

and, according to Eq. (25)

$$c_n\approx\exp\left[-\frac{D\left(n+i\frac{a}{D}\right)^2}{2b}\right]. \qquad (28)$$

Let us solve now Eq. (24) by the transition from a difference to a differential equation, which gives

$$b\frac{\partial c_n}{\partial t}=-(Dn+ia)c_n. \qquad (29)$$

As is easy to see, the solution of Eq. (29) coincides with the asymptotic (for large n) form (28) of exact solution.

After checking the applicability of our approximation by comparing with the known exact solutions for the field-free and steady-state cases, we proceed in the next Section to the analysis of the general Eq. (11), for which the exact solution is unknown.

4. Overdamped pendulum

The calculations to be described here are similar to those performed in Section 2. Consider Eq. (11), where a and b might be arbitrary functions of t

$$\frac{\partial c_n}{\partial t}+b(t)n\frac{\partial c_n}{\partial n}=\left[-Dn^2-ina(t)\right]c_n. \qquad (30)$$

The characteristic equations associated with Eq. (30) have the following form

$$\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}n}{nb(t)}=-\frac{\mathrm{d}c_n}{Dn^2+ina(t)}. \qquad (31)$$

The first equation in (31) defines the first constant of integration

$$C_1=n\exp\left[-\int_0^t b(\tau)\mathrm{d}\tau\right] \qquad (32)$$

while from the second equation, one gets

$$c_n=C_2\left\{n\exp\left[-\int_0^t b(\tau)\mathrm{d}\tau\right]\right\}\times$$
$$\exp\left[-DC_1^2\int_0^t \mathrm{d}z\exp\left(2\int_0^z b(\tau)\right)-\right.$$
$$\left.iC_1\int_0^t \mathrm{d}za(z)\exp\left(\int_0^z b(\tau)\mathrm{d}\tau\right)\right]. \qquad (33)$$

Taking Eq. (33) into account, according to the theory of partial differential equations, the second constant of integration

$$c_n=\exp\left[-in\exp\left(-\int_0^t b(\tau)\mathrm{d}\tau\right)\theta_0\right]\times$$
$$\exp\left[-Dn^2\exp\left(-2\int_0^t b(\tau)\mathrm{d}\tau\right)\left(\int_0^t \mathrm{d}z\exp\left(2\int_0^z b(\tau)\mathrm{d}\tau\right)\right)-\right.$$
$$\left.in\exp\left[\left(-\int_0^t b(\tau)\mathrm{d}\tau\right)\left(\int_0^t za(z)\exp\left(\int_0^z b(\tau)\mathrm{d}\tau\right)\right)\right]\right]. \qquad (34)$$

For the special case of time-independent a and b, Eq. (34) reduces to

$$c_n=\exp\left[-in\theta_0\exp(-bt)\right]\times$$
$$\exp\left\{-\frac{Dn^2}{2b}\left[1+\exp(-2bt)\right]-\frac{ina}{b}\left[1-\exp(-bt)\right]\right\}. \qquad (35)$$

It is a matter of direct verification to confirm that Eqs. (34) and (35) are solutions of Eq. (30) with $a=a(t)$, $b=b(t)$ and $a=\mathrm{const}$, $b=\mathrm{const}$, respectively.

Substituting Eq. (35) into Eq. (4), one obtains the full solution of the Fokker–Planck equation (8) corresponding to the Langevin equation (7), while Eq. (34) relates to the Langevin equation (7) with the time-dependent coefficient. Recall that all these solutions have been obtained on the assumption that one can replace the difference in n tridiagonal recurrence relations by differential equations. Now we are in a position to understand the final results of this assumption. It turns out that our final result (35) coincides with the solution of the Ornstein-Uhlenbeck process ($\sin(\theta)\to\theta$ in the original equation) which can be found in the book by Gardiner [10] for $a = 0$. Therefore, our approximation describes the limiting case of small angles, as we have already seen in the field-free case described in Section 2.

Provided the distribution function $P(\theta,t)$ is known, one can find all the correlation functions. For our case, which reduces to the Ornstein-Uhlenbeck approximation, the correlation functions are: $<\theta(t)\theta(t_1)>$ and $<\cos[\theta(t)]\cos[\theta(t_1)]>$, which have been calculated in [10] and [7], respectively.

5. Odd potential

In two previous sections, we considered the specific form of the periodic function $g(\theta)=\sin\theta$, which is important for many applications. The general case of the odd function $g(\theta)$ is described by Eq. (13) which looks exactly like Eq. (30) upon replacing b by ibK_1 and a by $a+K_0$. Of course, although these equations look similar, the accuracy of our approximation strongly depends on the form of $g(\theta)$ since one assumes $n>k_{\max}$.

As an example, let us consider the special case of $g(\theta)=\sin^3(\theta)$, which, in contrast to

$g(\theta)=\sin(\theta)$, does not reduce to Ornstein-Uhlenbeck equation, and cannot be solved by expansion in θ. Since this function is odd, $K_0=0$. Using the well-known relation [9]

$$\sin^3(\theta)=\frac{1}{4}\left[-\sin(3\theta)+3\sin\theta\right] \qquad (36)$$

one concludes that $d_3=-d_{-3}=-\frac{1}{8i}$ and $d_1=-d_{-1}=\frac{3}{8i}$, so that $K_1=\frac{1}{i}$. Since $iK_1=1$, the final Eqs. (34) and (35) obtained in the previous Section for $g(\theta)=\sin(\theta)$ and $n>1$ also apply for $g(\theta)=\sin^3(\theta)$ for $n>3$.

6. Even potential

The case of an even potential is described by Eq. (14), which can be rewritten, after the separation of variables $c_n(n,t)=\exp(-\lambda t)q_n(n)$ as

$$\frac{2i}{K_2}\frac{\mathrm{d}^2q}{\mathrm{d}n^2}+\left[\frac{\lambda}{n}+i(a+nK_0)+Dn\right]q_n=0. \qquad (37)$$

The solution of Eq. (37) for the steady-state case, $\lambda=0$, presents no problem since Eq. (37) reduces to the differential equation for the Airy function [11].

In the general case $\lambda\neq 0$, one has to perform substitution $r(n)=\frac{\mathrm{d}}{\mathrm{d}n}\ln[q(n)]$ which transforms Eq. (37) into the Ricatti equation for $r(n)$ of the form

$$\frac{\mathrm{d}r}{\mathrm{d}n}+r^2=\frac{K_2}{2i}\left[\frac{\lambda}{n}+i(a+nK_0)+Dn\right], \qquad (38)$$

after which one obtains the required solution of Eq. (37),

$$\frac{\mathrm{d}q}{\mathrm{d}n}-r(n)=\exp\left(-\int r(n)\mathrm{d}n\right). \qquad (39)$$

In general, the solutions of Eqs. (38) and (39) cannot be obtained by quadratures, and one has to use the approximate or numeral methods.

For $g(\theta)=\cos^2\theta$, $d_2=-d_2=2d_0=1/4$, so that in this case, $K_0=1/2$ and $K_2=2$. To find the approximate solution of Fokker-Planck equation (3) for arbitrary even periodic function $g(\theta)$ with a finite numbers of terms d_k in its Fourier series (5) (till some $k_{\max}$), one has to find these d_k, and calculate two numbers $K_0=\sum_{k=-k_{\max}}^{k_{\max}}d_k$, and $K_2=\sum_{k=-k_{\max}}^{k_{\max}}k^2d_k$ in Eq. (37). The latter equation is justified for $n>k_{\max}$.

7. Conclusions

We have used the transition from difference to differential equations as an approximate method of solving tridiagonal recurrence equations. This well-known method of replacing a discrete integer variable *n* by a continuous variable has a relative error of order of n^{-2}, i. e. this approximation is justified for large *n*, and the error is decreased with *n*. According to expansion (4) in the Fourier series, only small θ make an essential contribution to this series for large *n*, compared with rapidly oscillating large terms. It is no wonder, therefore, that for the overdamped pendulum without periodic and random forces (Section 2), and in the presence of these forces in the steady-state (Section 3) and in the general time-dependent case (Section 4), our method coincides with the exact solution in the limit of small θ. The periodic force acting on the pendulum has a simple form $g(\theta)=b\sin\theta$, and after replacing sinθ by θ in the Fokker-Planck equation, the latter takes the Ornstein-Uhlenbeck form which allows an exact solution. However, for a slightly different form of a periodic force, say $g(\theta)=\sin^{2m-1}(\theta)$, the expansion for small θ has the form $\sin^{2m-1}(\theta)\approx\theta^{2m-1}$, and the appropriate Fokker-Planck equation cannot be solved. However, our method can be applied for arbitrary function $g(\theta)$, leading to the first-order partial differential Eq.(13) for odd functions $g(\theta)$ and to the second-order partial differential Eq. (14) for even functions $g(\theta)$. The coefficients in these equations contain a few simple combinations of the coefficients in the Fourier expansions of $g(\theta)$ under the assumption that these expansions contain a finite number $k_{\max}$ of terms, and the equations can easily be solved, as we illustrated by a few simple examples in Sections 5 and 6. Strictly speaking, our procedure is applicable for $n > k_{\max}$, and it has a relative error of order n^{-2}. There are different ways to improve the accuracy of our method. One way is to restrict this procedure to the values of n that are too small, $n \ge n_{\min}$, where the accuracy is high ($\approx n_{\min}^{-2}$), and the coefficients $c_1 ... c_n$ with $n < n_{\min}$ will be found from the appropriate tridiagonal recurrence relations with the previously found $c_{n_{\min}}$.

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Анализ тридиагональных рекуррентных соотношений в континуальном приближении

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Переход от разностного к дифференциальному уравнению позволяет решить тридиагональные рекуррентные соотношения, которые возникают, в частности, при анализе вращения броуновской частицы с трением при наличии периодической силы. Замена дискретных индексов в разложениях Фурье непрерывными оправдан для больших номеров, т. е. для малых углов. В простейшем случае синусоидальной силы наше решение действительно совпадает с решением, полученным путем разложения синуса в первоначальном уравнении Фоккера-Планка (предел Орнштейна-Уленбека). Однако уже в случае несколько более сложного потенциала разложение при малых углах не делает соответствующее уравнение Фоккера-Планка разрешимым. В то же время предлагаемый метод позволяет решить задачу для всех периодических потенциалов, для которых ряды Фурье содержат конечное число слагаемых типа $\sin^m(\theta)$ или $\cos^m(\theta)$. Четные либо нечетные функции требуют несколько различного подхода и рассматриваются отдельно.

Аналіз тридіагональних рекурентних співвідношень у континуальному наближенні

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Перехід від різницевого до диференціального рівняння дозволяє вирішити тридіагональні рекурентні співвідношення, які виникають, зокрема, при аналізі обертання броунівської частинки з тертям у присутності періодичної сили. Заміна дискретних індексів у розкладанні Фур'є неперервними виправдана для великих номерів, тобто для малих кутів. У найпростішому випадку синусоїдальної сили наше рішення співпадає із рішенням, отриманим шляхом розкладання синуса у початковому рівнянні Фоккера-Планка (границя Орнштейна-Уленбека). Однак уже у випадку дещо складнішого потенціалу розкладання при малих кутах не робить відповідне рівняння Фоккера-Планка вирішуваним. Водночас запропонований метод дозволяє вирішити задачу для всіх періодичних потенціалів, для яких ряди Фур'є містять кінцеву кількість доданків, типу $\sin^m(\theta)$ або $\cos^m(\theta)$. Парні чи непарні функції вимагають дещо іншого підходу і розглядаються окремо.